PHYS 4073 - Quantum Mechanics- Test 2 - Fall 2010

All problems are worth 25 points. Turn in solutions to four of the six problems to be graded. If you turn in more than four solutions, I will grade the first four. You are allowed to drop one-half of a test, so I will take the first two problems turned in as the first half-test and the second two problems turned in as the second half-test.

1 The Hamiltonian of a two-state system in the basis $\{|1>, |2>\}$ is

$$\hat{H} = \hbar\omega \begin{pmatrix} 1 & -2i \\ 2i & 1 \end{pmatrix}$$

The system is in the state

$$|\psi> = \frac{1}{\sqrt{2}}(|1> + |2>)$$

What energies could be measured for the system in this state with what probability? What is the expectation value of the energy?

2 A system is in a simple gravitation potential $V(x) = -mgx$. Compute the uncertainty relation for the energy of the system and the position when the system is in a general state $|\psi>$. Compute the uncertainty relation for the energy of the system and the momentum when the system is in a general state $|\psi>$.

3 Show that a system moving in a simple gravitation potential $V(x) = -mgx$ has the correct classical behavior, that is show the time dependance of the average position and momentum is what you would expect classically.

4 Consider two operators of a two-state system in the basis $\{|1>, |2>\}$

$$\hat{A} = a \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

and

$$\hat{B} = b \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

Which operator or operators could represent a physical observable and why? Find the eigenvalues of both matrices. Were they what you expected? Comment. Calculate $[\hat{A}, \hat{B}]$.

5 For the space spanned by the lowest three energy states of the simple harmonic oscillator, $\{|\hbar\omega/2>, |3\hbar\omega/2>, |5\hbar\omega/2>\}$ write the matrix representing the operator

$$\hat{A} = \hat{a}_+^2 + \hat{a}_-^2$$

where $\hat{a}_+$ and $\hat{a}_-$ are the raising and lowering operators. Is $\hat{A}$ Hermitian? Justify.

6 Consider the three lowest energy states, $\{|1>, |2>, |3>\}$ of an infinite square well with $V = \infty$ outside the range 0 to a. Write the Hamiltonian matrix and the matrix representing the position operator $\hat{X}$.

1

$$\hat{H} = \hbar\omega \begin{pmatrix} 1 & -2i \\ 2i & 1 \end{pmatrix}$$

<u>Find eigenvalues</u>

$$\det(\hat{H} - \lambda \hat{1}) = 0$$

$$= \hbar\omega \begin{vmatrix} 1-\lambda & -2i \\ 2i & 1-\lambda \end{vmatrix}$$

$$= \hbar\omega \left[(1-\lambda)^2 - 4 \right]$$

$$= \hbar\omega (\lambda^2 - 2\lambda - 3)$$

$$= \hbar\omega (\lambda - 3)(\lambda + 1)$$

<u>Eigenvalues</u>

$$\lambda = -\hbar\omega, \; 3\hbar\omega$$

<u>Eigenvectors</u>

$$\lambda = -\hbar\omega$$

$$\begin{pmatrix} 2 & -2i \\ 2i & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \qquad 2a_1 - 2i a_2 = 0$$

$$a_1 = 1, \quad a_2 = -i$$

$$|-\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{\sqrt{2}}|1\rangle - \frac{i}{\sqrt{2}}|2\rangle$$

$\underline{\lambda = 3}$

$$\begin{pmatrix}-2 & -2i\\ 2i & -2\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = 0 \Rightarrow \quad \begin{array}{l} -2a_1 - 2ia_2 = 0 \\ a_1 = 1, \; a_2 = i \end{array}$$

$$|3\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}|1\rangle + \frac{i}{\sqrt{2}}|2\rangle$$

$$\langle -\hbar\omega|3\hbar\omega\rangle = \frac{1}{2}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = 0 \quad \checkmark$$

<u>Write $|\psi\rangle$ in terms of eigenvectors of $\hat{H}$</u>

$$|\psi\rangle \overset{c}{=} \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle = c_1|-\hbar\omega\rangle + c_2|3\hbar\omega\rangle$$

$$c_1 = \langle -\hbar\omega|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}(1+i)$$

$$c_2 = \langle 3\hbar\omega|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}(1-i)$$

observation	prob
$-\hbar\omega$	$c_1^* c_1 = \frac{1}{4}(1+i)(1-i) = \frac{1}{2}$
$3\hbar\omega$	$c_2^* c_2 = \frac{1}{4}(1+i)(1-i) = \frac{1}{2}$

Expectation Value

$$\langle E \rangle = \frac{1}{2}(-\hbar\omega) + \frac{1}{2}(3\hbar\omega) = \hbar\omega$$

Alternate Forms Eigenvectors

$$|-\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = i\,|3\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$|+3\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = i\,|3\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ -1 \end{pmatrix}$$

$$= \cancel{\frac{1}{\sqrt{2}}} \quad -i\,|3\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

eigenvectors {{1, -2i}, {2i, 1}}

Assuming i is the imaginary unit | Use ... instead

Input:

$$\text{Eigenvectors}\left[\begin{pmatrix} 1 & -2i \\ 2i & 1 \end{pmatrix}\right]$$

Result:

$v_1 = \{-i, 1\}$

$v_2 = \{i, 1\}$

Corresponding eigenvalues:

$\lambda_1 = 3$

$\lambda_2 = -1$

New to Wolfram|Alpha?

A few things to try:

enter any date (e.g. a birth date)

enter any city (e.g. a home town)

enter any two stocks

enter any calculation

enter any math formula

more »

Bookm...

Related Links

(2) $V(x) = \cancel{mgx} \; -mgx \qquad \hat{H} = \frac{\hat{p}^2}{2m} - g\hat{x}$

$$\sigma_E^2 \sigma_x^2 \geq \left(\frac{1}{2i} \langle \psi | [\hat{H}, \hat{x}] | \psi \rangle \right)^2$$

$$[\hat{H}, \hat{x}] = \left[\frac{\hat{p}^2}{2m} + V(\hat{x}), \hat{x} \right] = \frac{1}{2m} [\hat{p}^2, \hat{x}]$$

$$= \frac{1}{2m} \left(\hat{p} \underbrace{[\hat{p}, \hat{x}]}_{-i\hbar} + \underbrace{[\hat{p}, \hat{x}]}_{-i\hbar} \hat{p} \right)$$

$$= \frac{-i\hbar}{m} \hat{p}$$

$$\sigma_E^2 \sigma_x^2 \geq \left(\frac{1}{2i} \cdot \frac{-i\hbar}{m} \langle \psi | \hat{p} | \psi \rangle \right)^2$$

$$\frac{1}{4} \frac{\hbar^2}{m^2} \langle \hat{p} \rangle^2$$

$$\sigma_E^2 \sigma_p^2 \geq \left(\frac{1}{2i} \langle [\hat{H}, \hat{p}] \rangle \right)^2$$

$$\sigma_E^2 \sigma_p^2 \geq \left(\frac{1}{2i} \langle [\hat{H}, \hat{P}] \rangle \right)^2$$

$$[\hat{H}, \hat{P}] = \left[\frac{\hat{P}^2}{2m} + V(\hat{x}), \hat{P} \right]$$

$$= [V(\hat{x}), \hat{P}] = -mg[\hat{x}, \hat{P}]$$

$$= -i\hbar gm$$

$$\sigma_E^2 \sigma_p^2 \geq \frac{\hbar^2}{4} m^2 g^2$$

③ Ehrenfest's Thm

$$\frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{x}]|\psi\rangle + 0$$

$$= \frac{i}{\hbar}\cdot\frac{-i\hbar}{m} \langle\psi|\hat{p}|\psi\rangle$$ From last problem

$$= \frac{\langle p\rangle}{m} = \text{velocity}$$ ✓

$$\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{p}]|\psi\rangle$$

$$= \frac{i}{\hbar}\cdot -i\hbar g = g$$

$V(x) = -gx \rightarrow$ force in $+g$ direction ✓

(4) Only $\hat{B}$ could represent a physical observable because only $\hat{B}$ is Hermitian.

<u>From Alpha</u>

Eigenvalues of $\hat{B}$

$$\lambda = \frac{b}{2}(5 \pm \sqrt{5})$$

<u>Eigenvalues of $\hat{A}$</u>

$$\lambda = a(1 \mp i)$$

The eigenvalues of $\hat{B}$ were real and the eigenvalues of $\hat{A}$ were not, as expected for Hermitian and non-Hermitian matrices.

$$\hat{A}\hat{B} = ab\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = ab\begin{pmatrix} 2+i & 1+3i \\ 2i+1 & i+3 \end{pmatrix}$$

$$BA = ab\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = ab\begin{pmatrix} 2+i & 2i+1 \\ 1+3i & i+3 \end{pmatrix}$$

Commutator

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$= ab \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Wolfram Alpha

eigenvectors {{1, i}, {i, 1}}

Input:

$$\text{Eigenvectors}\left[\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right]$$

Result:

$v_1 = \{1, 1\}$

$v_2 = \{-1, 1\}$

Corresponding eigenvalues:

$\lambda_1 = 1 + i$

$\lambda_2 = 1 - i$

eigenvectors {{2, 1}, {1, 3}}

Input:

$$\text{Eigenvectors}\left[\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\right]$$

Result:

$$v_1 = \left\{\frac{1}{2}\left(5 + \sqrt{5}\right) - 3,\ 1\right\}$$

$$v_2 = \left\{\frac{1}{2}\left(5 - \sqrt{5}\right) - 3,\ 1\right\}$$

Corresponding eigenvalues:

$$\lambda_1 = \frac{1}{2}\left(5 + \sqrt{5}\right)$$

$$\lambda_2 = \frac{1}{2}\left(5 - \sqrt{5}\right)$$

Alternate forms:

$$\begin{pmatrix} \frac{1}{2}\left(-1 + \sqrt{5}\right) & 1 \\ \frac{1}{2}\left(-1 - \sqrt{5}\right) & 1 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{5}}{2} & 1 \\ -\frac{1}{2} - \frac{\sqrt{5}}{2} & 1 \end{pmatrix}$$

(5) $|\phi_0\rangle = |\frac{\hbar\omega}{2}\rangle$

$|\phi_1\rangle = |\frac{3\hbar\omega}{2}\rangle$

$|\phi_2\rangle = |\frac{5\hbar\omega}{2}\rangle$

$$\hat{A} = \begin{pmatrix} \langle\phi_0|\hat{A}|\phi_0\rangle & \langle\phi_0|\hat{A}|\phi_1\rangle & \langle\phi_0|\hat{A}|\phi_2\rangle \\ \langle\phi_1|\hat{A}|\phi_0\rangle & \langle\phi_1|\hat{A}|\phi_1\rangle & \langle\phi_1|\hat{A}|\phi_2\rangle \\ \langle\phi_2|\hat{A}|\phi_0\rangle & \langle\phi_2|\hat{A}|\phi_1\rangle & \langle\phi_2|\hat{A}|\phi_2\rangle \end{pmatrix}$$

$$\hat{A}^{+} = (a_+^{2+}) + (\hat{a}_-^{2+}) = \hat{a}_-^2 + \hat{a}_+^2 = \hat{A}$$

so $\hat{A}$ is hermitian.

Only terms two elements apart in energy can be non-zero, so $\langle\phi_0|\hat{A}|\phi_2\rangle$ and $\langle\phi_2|\hat{A}|\phi_0\rangle$

$$\langle\phi_2|\hat{A}|\phi_0\rangle = \langle\phi_2|\hat{a}_+^2 + \hat{a}_-^2|\phi_0\rangle$$

$$= \langle\phi_2|\hat{a}_+^2|\phi_0\rangle = \sqrt{0+1}\,\langle\phi_2|\hat{a}_+|\phi_1\rangle$$

$$= \sqrt{0+1}\,\sqrt{1+1}\,\langle\phi_2|\phi_2\rangle = \sqrt{2}$$

Since Hermitian, $\langle \phi_0 | \hat{A} | \phi_2 \rangle = \sqrt{2}$

$$\hat{A} = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$

(6) The normalized infinite square well wave functions are

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

$$\langle \phi_1 | \hat{x} | \phi_1 \rangle = \langle \hat{x} \rangle = a/2 = \langle \phi_1 | \hat{x} | \phi_1 \rangle$$

$$= \langle \phi_3 | \hat{x} | \phi_3 \rangle$$

$$\langle \phi_1 | \hat{x} | \phi_2 \rangle = \frac{2}{a} \int_0^a dx \, x \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a}$$

$$= \frac{-16a}{9\pi^2}$$

$$\langle \phi_1 | \hat{x} | \phi_3 \rangle = \frac{2}{a} \int_0^a dx \, x \sin \frac{\pi x}{a} \sin \frac{3\pi x}{a}$$

$$= 0$$

$$\langle \phi_2 | \hat{x} | \phi_3 \rangle = \frac{2}{a} \int_0^a dx \, x \sin \frac{2\pi x}{a} \sin \frac{3\pi x}{a}$$

$$= -\frac{48a}{25\pi^2}$$

$$\hat{X} = \begin{pmatrix} \frac{a}{2} & -\frac{16a}{9\pi^2} & 0 \\ -\frac{16a}{9\pi^2} & \frac{a}{2} & -\frac{48a}{25\pi^2} \\ 0 & -\frac{48a}{25\pi^2} & \frac{a}{2} \end{pmatrix}$$

$$\hat{X} = \begin{pmatrix} \langle\phi_0|\hat{x}|\phi_0\rangle & \langle\phi_0|\hat{x}|\phi_1\rangle & \langle\phi_0|\hat{x}|\phi_2\rangle \\ \langle\phi_1|\hat{x}|\phi_0\rangle & \langle\phi_1|\hat{x}|\phi_1\rangle & \langle\phi_1|\hat{x}|\phi_2\rangle \\ \langle\phi_2|\hat{x}|\phi_0\rangle & \langle\phi_2|\hat{x}|\phi_1\rangle & \langle\phi_2|\hat{x}|\phi_2 \end{pmatrix}$$

Likewise for $\hat{H}$, but

$$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$$

$$E_n = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{a^2}$$

$$\hat{H} = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} = \frac{\hbar^2\pi^2}{2ma^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

(%i2) phi1:sqrt((2/a))*sin(%pi*x/a);

(%o2) $\sqrt{2}\sqrt{\frac{1}{a}}\sin\left(\frac{\pi x}{a}\right)$

(%i3) phi2:sqrt((2/a))*sin(2*%pi*x/a);

(%o3) $\sqrt{2}\sqrt{\frac{1}{a}}\sin\left(\frac{2\pi x}{a}\right)$

(%i4) phi3:sqrt((2/a))*sin(3*%pi*x/a);

(%o4) $\sqrt{2}\sqrt{\frac{1}{a}}\sin\left(\frac{3\pi x}{a}\right)$

(%i5)

(%i6) assume(a>0);

(%o6) $[a>0]$

(%i7) integrate(x*phi1*phi2, x, 0, a);
integrate(x*phi1*phi3, x, 0, a);
integrate(x*phi2*phi3, x, 0, a);

(%o7) $-\frac{16a}{9\pi^2}$

(%o8) 0

(%o9) $-\frac{48a}{25\pi^2}$