

PHYS 4073 - Quantum Mechanics- Test 2 - Fall 2010

All problems are worth 25 points. Turn in solutions to four of the six problems to be graded. If you turn in more than four solutions, I will grade the first four. You are allowed to drop one-half of a test, so I will take the first two problems turned in as the first half-test and the second two problems turned in as the second half-test.

- 1 The Hamiltonian of a two-state system in the basis $\{|1\rangle, |2\rangle\}$ is

$$\hat{H} = \hbar\omega \begin{pmatrix} 1 & -2i \\ 2i & 1 \end{pmatrix}$$

The system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$

What energies could be measured for the system in this state with what probability? What is the expectation value of the energy?

- 2 A system is in a simple gravitation potential $V(x) = -mgx$. Compute the uncertainty relation for the energy of the system and the position when the system is in a general state $|\psi\rangle$. Compute the uncertainty relation for the energy of the system and the momentum when the system is in a general state $|\psi\rangle$.

- 3 Show that a system moving in a simple gravitation potential $V(x) = -mgx$ has the correct classical behavior, that is show the time dependence of the average position and momentum is what you would expect classically.

- 4 Consider two operators of a two-state system in the basis $\{|1\rangle, |2\rangle\}$

$$\hat{A} = a \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

and

$$\hat{B} = b \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

Which operator or operators could represent a physical observable and why? Find the eigenvalues of both matrices. Were they what you expected? Comment. Calculate $[\hat{A}, \hat{B}]$.

- 5 For the space spanned by the lowest three energy states of the simple harmonic oscillator, $\{|\hbar\omega/2\rangle, |3\hbar\omega/2\rangle, |5\hbar\omega/2\rangle\}$ write the matrix representing the operator

$$\hat{A} = \hat{a}_+^2 + \hat{a}_-^2$$

where $\hat{a}_+$ and $\hat{a}_-$ are the raising and lowering operators. Is $\hat{A}$ Hermitian? Justify.

- 6 Consider the three lowest energy states, $\{|1\rangle, |2\rangle, |3\rangle\}$ of an infinite square well with $V = \infty$ outside the range 0 to a . Write the Hamiltonian matrix and the matrix representing the position operator $\hat{X}$.

①

$$\hat{H} = \hbar\omega \begin{pmatrix} 1 & -2i \\ 2i & 1 \end{pmatrix}$$

Find eigenvalues

$$\det(\hat{H} - \lambda \hat{I}) = 0$$

$$= \hbar\omega \begin{vmatrix} 1-\lambda & -2i \\ 2i & 1-\lambda \end{vmatrix}$$

$$= \hbar\omega [(1-\lambda)^2 - 4]$$

$$= \hbar\omega (\lambda^2 - 2\lambda - 3)$$

$$\cancel{\hbar\omega} = \hbar\omega (\lambda - 3)(\lambda + 1)$$

Eigenvalues

$$\lambda = -\hbar\omega, 3\hbar\omega$$

Eigenvectors

$$\lambda = -\hbar\omega$$

$$\begin{pmatrix} 2 & -2i \\ 2i & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\cancel{2}a_1 - 2ia_2 = 0$$

$$a_1 = 1, a_2 = -i$$

$$|-hw\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}} |1\rangle - \frac{i}{\sqrt{2}} |2\rangle$$

$$\underline{\lambda = 3}$$

$$\begin{pmatrix} -2 & -2i \\ 2i & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \Rightarrow \begin{aligned} -2a_1 - 2ia_2 &= 0 \\ a_1 &= 1, a_2 = i \end{aligned}$$

$$|3hw\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} |1\rangle + \frac{i}{\sqrt{2}} |2\rangle$$

$$\langle -hw | 3hw \rangle = \frac{1}{2} (1 \ i) \begin{pmatrix} 1 \\ i \end{pmatrix} = 0 \quad \checkmark$$

Write $|\psi\rangle$ in terms of eigenvectors of $\hat{H}$

$$|\psi\rangle = \frac{1}{\sqrt{2}} |1\rangle + \frac{1}{\sqrt{2}} |2\rangle = c_1 |-hw\rangle + c_2 |3hw\rangle$$

$$c_1 = \langle -hw | \psi \rangle = \frac{1}{\sqrt{2}} (1 \ i) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} (1+i)$$

$$c_2 = \langle 3hw | \psi \rangle = \frac{1}{\sqrt{2}} (1-i) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} (1-i)$$

observation	prob
$-\hbar\omega$	$c_1^* c_1 = \frac{1}{4} (1+i)(1-i) = \frac{1}{2}$
$3\hbar\omega$	$c_2^* c_2 = \frac{1}{4} (1+i)(1-i) = \frac{1}{2}$

Expectation Value

$$\langle E \rangle = \frac{1}{2} (-\hbar\omega) + \frac{1}{2} (3\hbar\omega) = \hbar\omega$$

Alternate Forms Eigenvectors

$$|-\pi\omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = i | \pi\omega \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$|+3\pi\omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = i | 3\pi\omega \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} -i | 3\pi\omega \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

eigenvectors $\{\{1, -2\}, \{2i, 1\}\}$ Assuming i is the imaginary unit | Use $\sqrt{-1}$ or $\sqrt{-1}i$ instead

Input:

Eigenvalues $\left[\begin{pmatrix} 1 & -2i \\ 2i & 1 \end{pmatrix}\right]$

Result:

$$v_1 = \{-i, 1\}$$

$$v_2 = \{i, 1\}$$

Corresponding eigenvalues:

$$\lambda_1 = 3$$

$$\lambda_2 = -1$$

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$$(2) \quad V(x) = \cancel{mgx} - mgx \quad \hat{H} = \frac{\hat{p}^2}{2m} - g\hat{x}$$

$$\sigma_E^2 \sigma_x^2 \geq \left(\frac{1}{2i} \langle \psi | [\hat{H}, \hat{x}] | \psi \rangle \right)^2$$

$$[\hat{H}, \hat{x}] = \left[\frac{\hat{p}^2}{2m} + V(x), \hat{x} \right] = \frac{1}{2m} [\hat{p}^2, \hat{x}]$$

$$= \frac{1}{2m} \left(\underset{-i\hbar}{\hat{p}[\hat{p}, \hat{x}]} + \underset{-i\hbar}{[\hat{p}, \hat{x}]\hat{p}} \right)$$

$$= -\frac{i\hbar}{m} \hat{p}$$

$$\sigma_E^2 \sigma_x^2 \geq \left(\frac{1}{2i} \cdot -\frac{i\hbar}{m} \langle \psi | \hat{p} | \psi \rangle \right)^2$$

$$\frac{1}{4} \frac{\hbar^2}{m^2} \langle \hat{p} \rangle^2$$

$$\longrightarrow \sigma_E^2 \sigma_p^2 \geq \left(\frac{1}{2i} \langle [\hat{H}, \hat{p}] \rangle \right)^2$$

$$\sigma_E^2 \sigma_P^2 \geq \left(\frac{1}{2i} \langle [\hat{H}, \hat{P}] \rangle \right)^2$$

$$[\hat{H}, \hat{P}] = \left[\frac{\hat{P}^2}{2m} + V(\hat{x}), \hat{P} \right]$$

$$= [V(\hat{x}), \hat{P}] = -mg [\hat{x}, \hat{P}]$$

$$= -i\hbar gm$$

$$\sigma_E^2 \sigma_P^2 \geq \frac{\hbar^2}{4} m^2 g^2$$

③ Ehrenfest's Thm

$$\frac{d\langle x \rangle}{dt} = \frac{i}{\hbar} \langle \psi | [\hat{H}, \hat{x}] | \psi \rangle + 0$$

$$= \frac{i \cdot -i\hbar}{\hbar m} \langle \psi | \hat{p} | \psi \rangle$$

From last problem

$$= \frac{\langle p \rangle}{m} = \text{velocity} \quad \checkmark$$

$$\frac{d\langle p \rangle}{dt} = \frac{i}{\hbar} \langle \psi | [\hat{H}, \hat{p}] | \psi \rangle$$

$$= \frac{i}{\hbar} \cdot -i\hbar g = g$$

$$V(x) = -gx \rightarrow \text{force in } +g \text{ direction} \quad \checkmark$$

- ④ Only $\hat{B}$ could represent a physical observable because only $\hat{B}$ is Hermitian.

From Alpha

Eigenvalues of $\hat{B}$

$$\lambda = \frac{1}{2}(5 \pm \sqrt{5})$$

Eigenvalues of $\hat{A}$

$$\lambda = a(1 \mp i)$$

The eigenvalues of $\hat{B}$ were real and the eigenvalues of $\hat{A}$ were not, as expected for Hermitian and non-Hermitian matrices.

$$\hat{A}\hat{B} = ab \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = ab \begin{pmatrix} 2+i & 1+3i \\ 2i+1 & i+3 \end{pmatrix}$$

$$\hat{B}\hat{A} = ab \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = ab \begin{pmatrix} 2+i & 2i+1 \\ 1+3i & i+3 \end{pmatrix}$$

Commutator

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$= ab \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$



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eigenvectors $\{\{1, i\}, \{1, 1\}\}$



In[1]:

Eigenvectors $\left[\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right]$

Result:

$$v_1 = \{1, 1\}$$

$$v_2 = \{-1, 1\}$$

Corresponding eigenvalues:

$$\lambda_1 = 1 + i$$

$$\lambda_2 = 1 - i$$

eigenvectors {{2, 1}, {1, 3}}

Input:

 $\text{Eigenvectors}\left[\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\right]$

Result:

$$v_1 = \left\{ \frac{1}{2} (5 + \sqrt{5}) - 3, 1 \right\}$$

$$v_2 = \left\{ \frac{1}{2} (5 - \sqrt{5}) - 3, 1 \right\}$$

Corresponding eigenvalues:

$$\lambda_1 = \frac{1}{2} (5 + \sqrt{5})$$

$$\lambda_2 = \frac{1}{2} (5 - \sqrt{5})$$

Alternate forms:

$$\begin{pmatrix} \frac{1}{2} (-1 + \sqrt{5}) & -1 \\ \frac{1}{2} (-1 - \sqrt{5}) & -1 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{5}}{2} & 1 \\ -\frac{1}{2} - \frac{\sqrt{5}}{2} & 1 \end{pmatrix}$$

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$$\textcircled{5} \quad |\phi_0\rangle = \left| \frac{\hbar\omega}{2} \right\rangle$$

$$|\phi_1\rangle = \left| \frac{3\hbar\omega}{2} \right\rangle$$

$$|\phi_2\rangle = \left| \frac{5\hbar\omega}{2} \right\rangle$$

$$\hat{A} = \begin{pmatrix} \langle \phi_0 | \hat{A} | \phi_0 \rangle & \langle \phi_0 | \hat{A} | \phi_1 \rangle & \langle \phi_0 | \hat{A} | \phi_2 \rangle \\ \langle \phi_1 | \hat{A} | \phi_0 \rangle & \langle \phi_1 | \hat{A} | \phi_1 \rangle & \langle \phi_1 | \hat{A} | \phi_2 \rangle \\ \langle \phi_2 | \hat{A} | \phi_0 \rangle & \langle \phi_2 | \hat{A} | \phi_1 \rangle & \langle \phi_2 | \hat{A} | \phi_2 \rangle \end{pmatrix}$$

$$\hat{A}^\dagger = (\hat{a}_+^{\dagger 2}) + (\hat{a}_-^{\dagger 2}) = \hat{a}_-^2 + \hat{a}_+^2 = \hat{A}$$

so $\hat{A}$ is hermitian.

Only terms two elements apart in energy can be non-zero, so $\langle \phi_0 | \hat{A} | \phi_2 \rangle$ and $\langle \phi_2 | \hat{A} | \phi_0 \rangle$

$$\langle \phi_2 | \hat{A} | \phi_0 \rangle = \langle \phi_2 | \hat{a}_+^2 + \hat{a}_-^2 | \phi_0 \rangle$$

$$= \langle \phi_2 | \hat{a}_+^2 | \phi_0 \rangle = \sqrt{0+1} \langle \phi_2 | \hat{a}_+ | \phi_1 \rangle$$

$$= \sqrt{0+1} \sqrt{1+1} \langle \phi_2 | \phi_2 \rangle = \sqrt{2}$$

Since Hermitian, $\langle \phi_0 | \hat{A} | \phi_2 \rangle = \sqrt{2}$

$$\hat{A} = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$

⑥ The normalized infinite square well wave functions are

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

$$\begin{aligned}\langle \phi_1 | \hat{x} | \phi_1 \rangle &= \langle \hat{x} \rangle = a/2 = \langle \phi_1 | \hat{x} | \phi_1 \rangle \\ &= \langle \phi_3 | \hat{x} | \phi_3 \rangle\end{aligned}$$

$$\begin{aligned}\langle \phi_1 | \hat{x} | \phi_2 \rangle &= \frac{2}{a} \int_0^a dx \, x \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a} \\ &= \frac{-16a}{9\pi^2}\end{aligned}$$

$$\begin{aligned}\langle \phi_1 | \hat{x} | \phi_3 \rangle &= \frac{2}{a} \int_0^a dx \, x \sin \frac{\pi x}{a} \sin \frac{3\pi x}{a} \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle \phi_2 | \hat{x} | \phi_3 \rangle &= \frac{2}{a} \int_0^a dx \, x \sin \frac{2\pi x}{a} \sin \frac{3\pi x}{a} \\ &= \frac{-48a}{25\pi^2}\end{aligned}$$

$$\hat{X} = \begin{pmatrix} \frac{a}{2} & -\frac{16a}{9\pi^2} & 0 \\ -\frac{16a}{9\pi^2} & \frac{a}{2} & -\frac{48a}{25\pi^2} \\ 0 & -\frac{48a}{25\pi^2} & \frac{a}{2} \end{pmatrix}$$

$$\hat{X} = \begin{pmatrix} \langle \phi_0 | \hat{X} | \phi_0 \rangle & \langle \phi_0 | \hat{X} | \phi_1 \rangle & \langle \phi_0 | \hat{X} | \phi_2 \rangle \\ \langle \phi_1 | \hat{X} | \phi_0 \rangle & \langle \phi_1 | \hat{X} | \phi_1 \rangle & \langle \phi_1 | \hat{X} | \phi_2 \rangle \\ \langle \phi_2 | \hat{X} | \phi_0 \rangle & \langle \phi_2 | \hat{X} | \phi_1 \rangle & \langle \phi_2 | \hat{X} | \phi_2 \rangle \end{pmatrix}$$

Likewise for $\hat{H}$, but

$$\hat{H} |\phi_n\rangle = E_n |\phi_n\rangle$$

$$E_n = \frac{\hbar^2}{2m} \frac{n^2 \pi^2}{a^2}$$

$$\hat{H} = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} = \frac{\hbar^2 \pi^2}{2ma^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

✓ (%i2) phi1:sqrt((2/a))*sin(%pi*x/a);

(%o2) $\sqrt{2} \sqrt{a} \sin\left(\frac{\pi x}{a}\right)$

✓ (%i3) phi2:sqrt((2/a))*sin(2*%pi*x/a);

(%o3) $\sqrt{2} \sqrt{a} \sin\left(\frac{2 \pi x}{a}\right)$

✓ (%i4) phi3:sqrt((2/a))*sin(3*%pi*x/a);

(%o4) $\sqrt{2} \sqrt{a} \sin\left(\frac{3 \pi x}{a}\right)$

■ (%i5)

✓ (%i6) assume(a>0);

(%o6) $[a > 0]$

✓ (%i7) integrate(x*phi1*phi2, x, 0, a);
 integrate(x*phi1*phi3, x, 0, a);
 integrate(x*phi2*phi3, x, 0, a);

(%o7) $-\frac{16 a}{9 \pi^2}$

(%o8) 0

(%o9) $-\frac{48 a}{25 \pi^2}$