QM + Probability

All our predictions in QM will involve determining the set of outcomes of an experiment and assigning probabilities to each outcome.

Sometimes the outcomes will be a finite set, like the energy levels of a two state system $\{E_+, E_-\}$, sometimes an infinite but enemerable set, like the energies of an infinite square well, $\{E_1, E_2, E_3, \ldots\}$ and sometimes an infinite continuous set like the locations in the infinite square well $\{x \in [0, a]\}$.

For the finite or discrete case, we will assign a probability $\mathcal{P}_i$ to each possible measurement outcome

$$\mathcal{P}(E_+) = \mathcal{P}_+$$

$$\text{or } \mathcal{P}(E_5) = \mathcal{P}_5$$

For continuous variables, we will report a probability density $\mathcal{P}(x)$, the probability x is found between x, $x + dx$.

Naturally, we insist an experiment has some outcome, so

$$\sum_{\text{outcomes}} \mathcal{P}_i = 1 \quad \text{or} \quad \int_{-\infty}^{\infty} \mathcal{P}(x)\,dx = 1$$

We will rarely calculate $\mathcal{P}_i$ or $\mathcal{P}(x)$ directly, instead we will calculate the wavefunction $\psi(x)$ or the state vector $|\psi\rangle = c_1|E_1\rangle + c_2|E_2\rangle + c_3|E_3\rangle \ldots$

The probabilities are computed from $\psi(x)$ or $\{c_i\}$.

$$\mathcal{P}(x) = \psi^*(x)\,\psi(x)$$

$$\mathcal{P}_i = c_i^* c_i$$

The $*$ indicates complex conjugate. While all our predicted outcomes and probabilities will be real, the wave function and the coefficients of the state vector are complex.

The equations that allow us to predict $\psi(x)$ or $\{c_i\}$ are linear and therefore will not give us and overall magnitude for $\{c_i\}$ just the relative magnitude.

$\Longrightarrow$ Our computed $\psi(x)$ or $\{c_i\}$ will not obey

$$\sum_i P_i = \sum_i c_i^* c_i = 1$$

or

$$\int P(x)\,dx = \int \psi^* \psi\, dx = 1$$

$\Longrightarrow$ A key step to any QM calculation is to insist the probability is one.

<u>Normalization</u> - Multiplying a computed wave function or state vector by a constant so the probability sums to one.

For wave function, $\psi' = A\psi$

$$\int \psi'^* \psi' \, dx = 1 = A A^* \int \psi^* \psi \, dx$$

Assuming A is real, which it alway is

$$A = \frac{1}{\sqrt{\int_{-\infty}^{\infty} \psi^* \psi \, dx}}$$

For state vector,

$$|\psi'\rangle = A|\psi\rangle = \overbrace{A c_1}^{c_1'}|1\rangle + A c_2 |2\rangle \cdots$$

$$1 = \sum c_i'^* c_i' = A A^* \sum c_i^* c_i$$

$$A = \frac{1}{\sqrt{\sum c_i^* c_i}}$$

Normalization is crucial. If you are not given that the wave function is normalized, you have to check.

Assuming now that you have normalized ψ all our probability stuff carries through

$$\text{Average}(f(x)) \equiv \langle f(x) \rangle = \int f(x)\, P(x)\, dx$$
$$= \int f(x)\, \psi^* \psi\, dx$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x\, \psi^*(x)\, \psi(x)\, dx$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2\, \psi^*(x)\, \psi(x)\, dx$$

Standard Deviation (uncertainty in x)

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

Probability x will be observed in the range $x \in [b, c]$

$$\mathcal{P}(x \in [b,c]) = \int_b^c \mathcal{P}(x)\,dx = \int_b^c \psi^* \psi \, dx$$

<u>Discretely</u> For outcome set $\{E_i\}$

$$\langle E \rangle = \sum E_i \mathcal{P}_i = \sum E_i c_i^* c_i$$

$$\langle E^2 \rangle = \sum E_i^2 \mathcal{P}_i = \sum E_i c_i^* c_i$$

$$\sigma_E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}$$