Quantum Mechanics Fall 2003- Test Two

Due Wednesday at Beginning of Class

Problem 1 Consider a two-dimensional infinite square well such that $0 < x < a$ and $0 < y < a$ in both the x and y dimensions. The state of the system is

$$\psi(x,y) = N \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} \sin\frac{2\pi x}{a} \sin\frac{2\pi y}{a}$$

where N is a normalization constant. The Hamiltonian separated as $H = H_x + H_y$.

(a) If H is measured, what are the possible outcomes with what probability.

(b) Calculate the expectation value of the energy $< H >$.

(c) If H_x is measured, what are the outcomes and with what probability.

(d) If a measurement of H_x yields, $\pi^2\hbar^2/2ma^2$, what are the possible outcomes of a measurement of H_y and with what probability?

Problem 2 The lowest three orthonormal energy eigenstates of the the simple harmonic oscillator are

$$\phi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\eta^2/2}$$

$$\phi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{2}\eta e^{-\eta^2/2}$$

$$\phi_2 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2}}(2\eta^2 - 1)e^{-\eta^2/2}$$

where

$$\eta = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} x$$

The energy of a simple harmonic oscillator is

$$E_n = (n + \frac{1}{2})\hbar\omega$$

where $\omega = \sqrt{\frac{k}{m}}$. The system is in the state

$$\psi(x, t = 0) = \left(\frac{m\omega}{2\pi\hbar}\right)^{\frac{1}{4}} e^{-\eta^2/2} \frac{1}{\sqrt{2}}\left[1 + \sqrt{2}\eta\right]$$

(a) What energy values could be observed and with what probabilities? Hint you do not need to integrate to do this. Hint 2, write the ψ as an expansion in terms of the functions above.

(b) Calculate $\psi(x,t)$

(c) Construct the combination of raising an lowering operators representing $\hat{X}$.

(d) Use the operators in (d) to evaluate the expectation values $< X >$ and $< X^2 >$ in the ground state.

(e) Two identical fermions are placed in the oscillator. Report the ground state energy and wave function of this system.

Problem 3 Consider an isotopic harmonic oscillator with potential

$$V(x,y,z) = \frac{1}{2}k(x^2+y^2+z^2) = \frac{1}{2}kr^2$$

(a) Write the Hamiltonian. Separate the Hamiltonian into a radial and angular parts. Solve the angular part.

(b) Write the differential equation the radial part satisfies. Write both the $R(r)$ and $u(r)$ equations.

(c) Solve the radial equation for $\ell = 0$ and compute the ground state energy. Hint, compare the radial equation to the equation for a simple one dimensional harmonic oscillator. Careful, the radial function $R(r)$ must be finite so a solution may have to be discarded.

(d) Calculate the expectation value of r and r^2 in this ground state. Note this is a three dimensional problem.

(e) Suppose the wave function of the system is

$$\psi = NR(r)(\cos\theta + \sin\theta e^{-\phi}),$$

where N is a normalization and $R(r)$ is the solution to the radial equation. What values could be measured for L^2 and what values for L_z with what probabilities for each.

Problem 4 Construct from the recurance relation the ψ_{421} state of Hydrogen. You may look up the appropriate angular function. Calculate the energy, total angular momentum, and z component of angular momentum. Report each as numbers.