Mechanics Spring 2003 - Test 2

Problem 2.1 The eccentricity of Mercury is 0.206 and the semi-major axis is 0.387AU where 1AU$= 1.5 \times 10^{11}$m. The mass of the sun is 2×10^{30}kg.

5 (a) Compute the period of Mercury. (In years or seconds).

5 (b) Compute the distance from the sun at perihelion and aphelion.

15 (c) Compute the velocity at perihelion.

Problem 2.2 Consider the force $\vec{F} = -x\hat{i} - z\hat{j} + y\hat{k}$. A particle of mass m experiencing the force is released at the point (d, d, d).

5 (a) Does a potential function exist for this force? If yes, calculate the potential function. If no, demonstrate why the potential function does not exist.

5 (b) Find the component equations of motion for this force.

10 (c) Solve the component equation of motion that is solvable using techniques from this class using the initial condition given.

5 (d) Make progress on the other two equations.

Problem 2.3 A particle moves in a potential $V(r) = -\frac{\gamma}{r^2}$ where γ is a constant and r is the distance from the origin.

5 (a) What is conserved in the motion, linear momentum, angular momentum, or energy? More than one may be conserved.

5 (b) Calculate the force on the particle.

5 (c) Write the differential equation of the orbit.

5 (d) Assuming $2\gamma/m\ell^2 < 1$, solve the differential equation of the orbit. Use A and $\theta_0 = 0$ as your constants of integration.

5 (e) Compute $\theta(t)$. You may need to ask for an integral.

Problem 2.4 John Hubbard's dad owns a dump truck. John is driving the truck when an exploding salad incident happens (common in the west, you will have to ask Bernadette) causing the truck to go into a spin about its center of mass. The center of mass decelerates at a rate of $g/4$. The truck spins about the center of mass once per second(assume the rate is constant). As luck would have it, Caleb is sitting at the center of mass.

4 (a) What inertial forces act on Caleb? Express the forces as a ratio of the force to Caleb's weight mg. Report only the magnitude of the forces.

Josh Hess is sitting at the edge of the dump truck a distance of 3m from Caleb and thus from the center of rotation.

10 (b) What inertial forces act on Josh? Express the forces as a ratio of the force to Josh's weight mg. Report only the magnitude of the forces.

10 (c) Caleb tries the throw Josh a rope in a flat trajectory at velocity $v = 30$m/s. What inertial forces act on the rope? Express the forces as a ratio of the force to the rope's weight mg and as the distance from Caleb r. Report only the magnitude of the forces.

1 (d) What are the odds Caleb saves Josh's life?

Bonus Problem Write your own trivia question with answer. If I use the question on the homework 5 bonus points.

Problem 2.1 $\qquad \varepsilon = 0.206$

(a) $\quad \tau^2 = a^3$ if τ in years and a is in AU

$$T = (0.387)^{3/2} \text{ years}$$
$$= 0.241 \text{ years}$$
$$= 88 \text{ days} \checkmark$$
$$= 7.56\times10^{6} \text{ s}$$

(b) $\quad r_0 = a(1-\varepsilon) = 0.307\,\text{AU} = 4.61\times10^{10}\,\text{m}$

$r_1 = a(1+\varepsilon) = 0.467\,\text{AU} = \cancel{1.81\times10^{11}\,\text{m}}$ 7×10^{10} m

(c) $\quad \alpha = \frac{m\ell^2}{k} \qquad \sqrt{\frac{k\alpha}{m}} = \ell$

$$\ell = \sqrt{M_\odot G \alpha} \qquad \alpha = (1-\varepsilon^2)a$$

$$\ell = \sqrt{M_\odot G a(1-\varepsilon^2)} = r_0 v_0$$

<u>2.1(b)</u>

$$v_0 = \frac{l}{r_0} = \frac{\sqrt{M_\odot G a (1-\varepsilon^2)}}{r_0}$$

$$v_0 = 59 \text{ Km/s}$$

Problem 2.2

(a) The force is conservative, has a potential function

if $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & -z & +y \end{vmatrix}$$

$$= \left(-1-(+1)\right)\hat{\imath} - \hat{\jmath}\,0 + 0\,\hat{k}$$

$$= -2\hat{\imath}$$ so no potential function exists.

(b)

$$m\ddot{x} = -x \qquad (1)$$

$$m\ddot{y} = -z \qquad (2)$$

$$m\ddot{z} = +y \qquad (3)$$

(c) Equation (1) can be solved as a simple harmonic oscillator.

$$\ddot{x} + \frac{1}{m}x = 0 \qquad \omega_0 = \sqrt{\frac{1}{m}}$$

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

$x(0) = d \quad \Rightarrow \quad A = d$

$\dot{x}(t) = -A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t$

$\dot{x}(0) = 0 \quad \Rightarrow \quad B = 0$

$x(t) = d \cos\sqrt{\frac{1}{m}}\, t$

(d) Differentiate (3) twice and sub (2)

$\ddddot{z} = \frac{1}{m}\ddot{y} = -\frac{1}{m^2} z$

$\ddddot{z} + \frac{1}{m^2} z = 0$

Likewise

$\ddddot{y} + \frac{1}{m^2} y = 0$

At which point I stop.

$$\ddddot{y} + \gamma \cancel{y} = 0$$

$$(D^4 + \gamma)y = 0$$

$$(D^2 + \sqrt{\gamma}\, i)(D^2 - \sqrt{\gamma}\, i)\, y = 0$$

$$(D^2 + \sqrt{\gamma}\, e^{i\pi/2})(D^2 + \sqrt{\gamma}\, e^{-i\pi/2})\, y = 0$$

$$(D + \gamma^{1/4}\sqrt{e^{i\pi/2}}\; i)(D - \gamma^{1/4}\sqrt{e^{i\pi/2}}\; i) \cdots \quad y = 0$$

$$e^{i\pi/4}\, e^{i\pi/2}$$

$$(D + \gamma^{1/4} e^{i\,3\pi/4})(D + \gamma^{1/4} e^{i\pi/4})$$

$$A e^{-\gamma^{1/4} e^{i3\pi/4}\, t} + \cdots$$

$$e^{i\,3\pi/4} = \cos\frac{3\pi}{4} + i \sin 3\pi/4$$

$$A e^{-\gamma^{1/4}\left(\cos\frac{3\pi}{4} + i \sin 3\pi/4\right) t}$$

+ Growing

Decaying Oscillations

2.3 (a) For a central force, both energy and angular momentum is conserved.

(b) $$\vec{F} = -\frac{\partial V}{\partial r} = -(-\gamma)\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)$$

$$\vec{F} = \frac{-2\gamma}{r^3} = -2\gamma u^3 \qquad \left(u = \frac{1}{r}\right)$$

(c) $$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2} f(u^{-1})$$

$$= \frac{2\gamma}{ml^2} u$$

(d) $$\frac{d^2u}{d\theta^2} + \left(1 - \frac{2\gamma}{ml^2}\right) u = 0$$

Let $B^2 = \left(1 - \frac{2\gamma}{ml^2}\right)$

$$\frac{d^2u}{d\theta^2} + B^2 u = 0$$

$$u = A \cos B(\theta - \theta_0) = A \cos B\theta$$

since $\theta_0 = 0$.

$$r = \frac{1}{A \cos B\theta}$$

(e) $$\ell = r^2 \dot{\theta}$$

$$= \frac{1}{A^2 \cos^2 B\theta} \frac{d\theta}{dt}$$

$$\int_0^t \ell A^2 \, dt = \int_0^\theta \frac{d\theta}{\cos^2 B\theta} = \int_0^\theta \sec^2 B(\theta) \, d\theta$$

$$= \frac{1}{B} \int_0^{B\theta} \sec^2 u \, du$$

$u = B\theta$

$du = B \, d\theta$

$$= \frac{1}{B} \tan(u) \Big|_0^{B\theta}$$

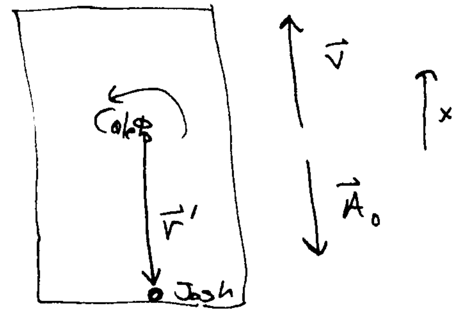
$$\ell A^2 t = \frac{1}{B} \tan B\theta$$

$$\theta(t) = \frac{1}{B} \tan^{-1} \left[B \ell A^2 t \right]$$

(2.4) Let center of mass be origin of moving frame.

(a) $A_0 = -g/4$

Coleb is stationary at the center of rotation.



The possible inertial forces are

$$-m\vec{A}_0 = -\frac{mg}{4}\hat{x}$$

$$-2m\vec{\omega}\times\vec{v}' \qquad \underline{\text{Coriolis}}$$

zero because Coleb is not moving.

$$-m\dot{\vec{\omega}}\times\vec{r}' \qquad \underline{\text{Transverse}}$$

zero because rate of rotation constant.

$$-m\vec{\omega}\times(\vec{\omega}\times\vec{r}') \qquad \underline{\text{Centrifugal}}$$

zero because $r' = 0$.

(b) So the total inertial force on Coleb is $|m\vec{A}_0|/mg = 1/4$

(b) Josh also is unmoving in the rotating frame but $\vec{r}\,' = 3m\hat{x} = d\hat{x}$. Josh still fells $|m\vec{A}_0| / mg = 1/4$, but also a centrifugal force

$$\vec{F}_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Let $\vec{\omega}$ be $\omega\hat{k}$

$$|\vec{F}_{cent}| = +\omega^2 d m$$

$$\left|\frac{F_{cent}}{mg}\right| = \frac{\omega^2 d}{g}$$

$$\omega = \frac{2\pi}{1s} = 2\pi(s^{-1})$$

$$\left|\frac{F_{cent}}{mg}\right| = \frac{4\pi^2 d}{g}(s^{-1})^2 = 12$$

Josh feels 12g, out of the truck he goes.

(c) The rope feels $g/4$ from the acceleration of the center of mass. ~~And~~

It also feels a centrifugal force,

$$\left|\frac{F_{cent}}{mg}\right| = \frac{4\pi^2 (s^{-1})^2 r}{g} = 4\,(m^{-1})\,r$$

The rope also feels a Coriolis force,

$$\left|\frac{F_{cor}}{mg}\right| = \frac{2}{g}\left|\vec{\omega} \times \vec{v}\,'\right| = \frac{2\omega v_0}{g}$$

$$= 38$$

a force of $38 g$

(d) Good try, but bye bye Josh.