Homework 7

Due Friday 3/28/2003 at beginning of class

Fowles Problems

- 7.1 6
- 7.2 6
- 7.8 6
- 7.9 3/6
- 7.12 6
- 7.14 10
- 7.16 6
- ~~7.21~~
- 7.25 8/2

86 pts.

Bonus Problem B.1 (Dominic from Car Talk) An engineer is driving a train. He lets it coast to a stop. He then tries to drive it again. The train is too heavy. The wheels spin and the train remains motionless. Sand doesn't help. What should he do?

+1

Back up the train

where m_0 is the initial mass of the rocket plus unburned fuel, m is the mass at any time, and V is the speed of the ejected fuel relative to the rocket. Owing to the nature of the logarithmic function, it is necessary to have a large fuel-to-payload ratio in order to attain the large speeds needed for satellite launching.

EXAMPLE 7.7.1

LAUNCHING AN EARTH SATELLITE FROM CAPE CANAVERAL

We know from Example 6.5.3 that the speed of a satellite in a circular orbit near Earth is about 8 km/s. Satellites are launched toward the east to take advantage of Earth's rotation. For a point on the Earth near the equator the rotational speed is approximately $R_{Earth}\omega_{Earth}$, which is about 0.5 km/s. For most rocket fuels the effective ejection speed is of the order of 2 to 4 km/s. For example, if we take $V = 2.5$ km/s, then we find that the mass ratio calculated from Equation 7.7.9 is

$$\frac{m_0}{m} = \exp\left(\frac{v - v_0}{V}\right) = \exp\left(\frac{8.0 - 0.5}{2.5}\right) = e^3 = 20.1$$

to achieve orbital speed from the ground. Thus, only about 5% of the total initial mass m_0 is payload.

PROBLEMS

7.1 A system consists of three particles, each of unit mass, with positions and velocities as follows:

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{i} + \mathbf{j} & \mathbf{v}_1 &= 2\mathbf{i} \\ \mathbf{r}_2 &= \mathbf{j} + \mathbf{k} & \mathbf{v}_2 &= \mathbf{j} \\ \mathbf{r}_3 &= \mathbf{k} & \mathbf{v}_3 &= \mathbf{i} + \mathbf{j} + \mathbf{k} \end{aligned}$$

Find the position and velocity of the center of mass. Find also the linear momentum of the system.

7.2 **(a)** Find the kinetic energy of the system in Problem 7.1.

(b) Find the value of $mv_{cm}^2/2$.

(c) Find the angular momentum about the origin.

7.3 A bullet of mass m is fired from a gun of mass M. If the gun can recoil freely and the muzzle velocity of the bullet (velocity relative to the gun as it leaves the barrel) is v_0, show that the actual velocity of the bullet relative to the ground is $v_0/(1 + \gamma)$ and the recoil velocity for the gun is $-\gamma v_0/(1 + \gamma)$, where $\gamma = m/M$.

7.4 A block of wood rests on a smooth horizontal table. A gun is fired horizontally at the block and the bullet passes through the block, emerging with half its initial speed just before it entered the block. Show that the fraction of the initial kinetic energy of the bullet that is lost as frictional heat is $\frac{3}{4} - \frac{1}{4}\gamma$, where γ is the ratio of the mass of the bullet to the mass of the block ($\gamma < 1$).

7.5 An artillery shell is fired at an angle of elevation of 60° with initial speed v_0. At the uppermost part of its trajectory, the shell bursts into two equal fragments, one of which moves directly upward, relative to the ground, with initial speed $v_0/2$. What is the direction and speed of the other fragment immediately after the burst?

7.6 A ball is dropped from a height h onto a horizontal pavement. If the coefficient of restitution is ϵ, show that the total vertical distance the ball goes before the rebounds cease is $h(1 + \epsilon^2)/(1 - \epsilon^2)$. Find also the total length of time that the ball bounces.

7.7 A small car of a mass m and initial speed v_0 collides head-on on an icy road with a truck of mass $4m$ going toward the car with initial speed $\frac{1}{2}v_0$. If the coefficient of restitution in the collision is $\frac{1}{4}$, find the speed and direction of each vehicle just after colliding.

7.8 Show that the kinetic energy of a two-particle system is $\frac{1}{2}mv_{cm}^2 + \frac{1}{2}\mu v^2$, where $m = m_1 + m_2$, v is the relative speed, and μ is the reduced mass.

7.9 If two bodies undergo a direct collision, show that the loss in kinetic energy is equal to

$$\frac{1}{2}\mu v^2(1 - \epsilon^2)$$

where μ is the reduced mass, v is the relative speed before impact, and ϵ is the coefficient of restitution.

7.10 A moving particle of mass m_1 collides elastically with a target particle of mass m_2 which is initially at rest. If the collision is head-on, show that the incident particle loses a fraction $4\mu/m$ of its original kinetic energy, where μ is the reduced mass and $m = m_1 + m_2$.

7.11 Show that the angular momentum of a two-particle system is

$$\mathbf{r}_{cm} \times m\mathbf{v}_{cm} + \mathbf{R} \times \mu\mathbf{v}$$

where $m = m_1 + m_2$, μ is the reduced mass, $\mathbf{R}$ is the relative position vector, and $\mathbf{v}$ is the relative velocity of the two particles.

7.12 The observed period of the binary system Cygnus X-1, presumed to be a bright star and a black hole, is 5.6 days. If the mass of the visible component is $20\,M_\odot$ and the black hole has a mass of $16\,M_\odot$, show that the semimajor axis of the orbit of the black hole relative to the visible star is roughly one fifth the distance from Earth to the Sun.

7.13 **(a)** Using the coordinate convention given in Section 7.4 for the restricted three-body problem, find the coordinates (x', y') of the two Lagrangian points, L_4 and L_5.

(b) Show that the gradient of the effective potential function $V(x', y')$ vanishes at L_4 and L_5.

7.14 A proton of mass m_p with initial velocity $\mathbf{v}_0$ collides with a helium atom, mass $4m_p$, that is initially at rest. If the proton leaves the point of impact at an angle of 45° with its original line of motion, find the final velocities of each particle. Assume that the collision is perfectly elastic.

7.15 Work Problem 7.14 for the case that the collision is inelastic and that Q is equal to one-fourth of the initial energy of the proton.

7.16 Referring to Problem 7.14, find the scattering angle of the proton in the center-of-mass system.

7.17 Find the scattering angle of the proton in the center-of-mass system for Problem 7.15.

7.18 A particle of mass m with initial momentum p_1 collides with a particle of equal mass at rest. If the magnitudes of the final momenta of the two particles are p_1' and p_2'

tively, show that the energy loss of the collision is given by

$$Q = \frac{p_1' p_2'}{m} \cos \psi$$

where ψ is the angle between the paths of the two particles after colliding.

7.19 A particle of mass m_1 with an initial kinetic energy T_1 makes an elastic collision with a particle of mass m_2 initially at rest. m_1 is deflected from its original direction with a kinetic energy T_1' through an angle ϕ_1 as in Figure 7.6.1. Letting $\alpha = m_2/m_1$ and $\gamma = \cos \phi_1$, show that the fractional kinetic energy lost by m_1, $\Delta T_1/T_1 = (T_1 - T_1')/T_1$, is given by

$$\frac{\Delta T_1}{T_1} = \frac{2}{1+\alpha} - \frac{2\gamma}{(1+\alpha)^2}\left(\gamma + \sqrt{\alpha^2 + \gamma^2 - 1}\right)$$

7.20 A particle of mass m_1 scatters elastically from a particle of mass m_2 initially at rest as described in Problem 7.19. Find the curve $r(\phi_1)$ such that the time it takes the scattered particle to travel from the collision point to any point along the curve is a constant.

7.21 A uniform chain lies in a heap on a table. If one end is raised vertically with uniform velocity v, show that the upward force that must be exerted on the end of the chain is equal to the weight of a length $z + (v^2/g)$ of the chain, where z is the length that has been uncoiled at any instant.

7.22 Find the differential equation of motion of a raindrop falling through a mist collecting mass as it falls. Assume that the drop remains spherical and that the rate of accretion is proportional to the cross-sectional area of the drop multiplied by the speed of fall. Show that if the drop starts from rest when it is infinitely small, then the acceleration is constant and equal to $g/7$.

7.23 A uniform heavy chain of length a hangs initially with a part of length b hanging over the edge of a table. The remaining part, of length $a - b$, is coiled up at the edge of the table. If the chain is released, show that the speed of the chain when the last link leaves the end of the table is $[2g(a^3 - b^3)/3a^2]^{1/2}$.

7.24 A balloon of mass M containing a bag of sand of mass m_0 is filled with hot air until it becomes buoyant enough to rise ever so slightly above the ground, where it then hovers in equilibrium. Sand is then released at a constant rate such that all of it is dumped out in a time t_0. Find **(a)** the height of the balloon and **(b)** its velocity when all the sand has been released. Assume that the upward buoyancy force remains constant and neglect air resistance. **(c)** Assume that $\epsilon = m_0/M$ is very small, and find a power series expansion of your solutions for parts (a) and (b) in terms of this ratio. **(d)** Letting $M = 500$ kg, $m_0 = 10$ kg, and $t_0 = 100$ s, and keeping only the first-order term in the expansions obtained in part (c), find a numerical value for the height and velocity attained when all the sand has been released.

7.25 A rocket, whose total mass is m_0, contains a quantity of fuel, whose mass is ϵm_0 $(0 < \epsilon < 1)$. Suppose that, on ignition, the fuel is burnt at a constant mass-rate k, ejecting gasses with a constant speed V relative to the rocket. Assume that the rocket is in a force-free environment.

(a) Find the distance that the rocket has traveled at the moment it has burnt all the fuel.

(b) What is the maximum possible distance that the rocket can travel during the burning phase? Assume that it starts from rest.

7.26 A rocket traveling through the atmosphere experiences a linear air resistance $-k\mathbf{v}$. Find the differential equation of motion when all other external forces are negligible. Integrate the equation and show that if the rocket starts from rest, the final speed is given by $v = V\alpha[1 - (m/m_0)^{1/\alpha}]$, where V is the relative speed of the exhaust fuel. $\alpha = |\dot{m}/k| =$ constant, m_0 is the initial mass of the rocket plus fuel, and m is the final mass of the rocket.

7.27 Find the equation of motion for a rocket fired vertically upward, assuming g is constant. Find the ratio of fuel to payload to achieve a final speed equal to the escape speed v_e from the Earth if the speed of the exhaust gas is kv_e, where k is a given constant, and the fuel burning rate is $|\dot{m}|$. Compute the numerical value of the fuel–payload ratio for $k = \frac{1}{4}$, and $|\dot{m}|$ equal to 1% of the mass of the fuel per second.

COMPUTER PROBLEMS

C 7.1 Let two particles ($m_1 = m_2 = 1$ kg) repel each other with equal and opposite forces given by

$$\mathbf{F}_{12} = k\frac{b^2}{r^2}\mathbf{r}_{12} = -\mathbf{F}_{21}$$

where $b = 1$ m and $k = 1$ N. Assume that the initial positions of m_1 and m_2 are given by $(x_1, y_1)_0 = (-10, 0.5)$ m and $(x_2, y_2)_0 = (0, -0.5)$ m. Let the initial velocity of m_1 be 10 m/s in the $+x$ direction and m_2 be at rest. Numerically integrate the equations of motion for these two particles undergoing this two-dimensional "collision."

(a) Plot their trajectories up to a point where their distance of separation is 10 m.

(b) Measure the scattering angle of the incident particle and the recoil angle of the scattered particle. Is the sum of these two angles equal to 90°?

(c) What is the vector sum of their final momenta? Is it equal to the initial momentum of the incident particle?

C 7.2 Using a numerical optimization tool such as *Mathematica*'s *FindMinimum* function, find the coordinates (x', y') of the Lagrange point L_4 in the restricted three-body problem. Do not assume, as you probably did in Problem 7.13, that L_4 is located at one of the corners of an equilateral triangle whose opposite base is formed by the two primaries. However, you should start the search for the coordinates of L_4 by using a point near the position of the suspected solution.

C 7.3 The total mass of a new experimental rocket, including payload, is 2×10^6 kg, and 90% of its mass is fuel. It burns fuel at a constant rate of 18,000 kg/s and exhausts the spent gasses at a speed of 3000 m/s. Assume that the rocket is launched vertically. Ignore the rotation of the Earth.

(a) Ignore air resistance, and assume that g, the acceleration due to gravity, is constant. Calculate the maximum altitude attained by the launched rocket.

(b) Repeat part (a), but include the effect of air resistance and the variation of g with altitude. Assume that the rocket presents a resistive surface to air that is equiva-

7.1

$$\vec{r}_1 = \hat{\imath} + \hat{\jmath} \qquad \vec{v}_1 = 2\hat{\imath}$$

$$\vec{r}_2 = \hat{\jmath} + \hat{k} \qquad \vec{v}_2 = \hat{\jmath}$$

$$\vec{r}_3 = \vec{k} \qquad \vec{v}_3 = \hat{\imath} + \hat{\jmath} + \hat{k}$$

$$m_i = 1 \qquad m = 3$$

$$\vec{r}_{cm} = \frac{1}{m} \sum m_i \vec{r}_i$$

$$= \frac{1}{3}\left[\hat{\imath} + \hat{\jmath} + \hat{\jmath} + \hat{k} + \hat{k}\right]$$

$$= \frac{1}{3}\left(\hat{\imath} + 2\hat{\jmath} + 2\hat{k}\right)$$

$$\vec{v}_{cm} = \dot{\vec{r}}_{cm} = \frac{1}{m} \sum m_i \vec{v}_i$$

$$= \frac{1}{3}\left(2\hat{\imath} + \hat{\jmath} + \hat{\imath} + \hat{\jmath} + \hat{k}\right)$$

$$= \frac{1}{3}\left(3\hat{\imath} + 2\hat{\jmath} + \hat{k}\right)$$

$$\vec{p} = \sum m_i \vec{v}_i = m\vec{v}_{cm} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$$

7.2 (a) $T = \sum \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i$

$$= \frac{1}{2}\left(4 + 1 + 3\right)$$

$$T = 4$$

(b) $\frac{1}{2} m v_{cm}^2 = \frac{1}{2} m \left(\frac{1}{3}\right)\left(3\hat{i} + 2\hat{j} + \hat{k}\right) \cdot \left(\frac{1}{3}\right)\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$

$$= \frac{m}{18}\left(9 + 4 + 1\right)$$

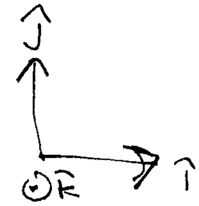
$$= \frac{14m}{18} = \frac{14}{6} = \frac{7}{3} = 2\tfrac{1}{3}$$

(c) $\vec{L} = \sum \vec{r}_i \times m_i \vec{v}_i$

$$= (\hat{i} + \hat{j}) \times (2\hat{i}) + (\hat{j} + \hat{k}) \times \hat{j} + \hat{k} \times (\hat{i} + \hat{j} + \hat{k})$$

$$= 2\hat{j} \times \hat{i} + 2\hat{k} \times \hat{j} + \hat{k} \times \hat{i}$$

$$= -2\hat{k} - 2\hat{i} + \hat{j}$$



7.8

$$v \equiv v_1 - v_2 \qquad m = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{m}$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$r_{cm} = \frac{1}{m}\left[m_1 r_1 + m_2 r_2\right]$$

$$v_{cm} = \dot{r}_{cm} = \frac{1}{m}\left[m_1 v_1 + m_2 v_2\right]$$

$$\frac{1}{2} m v_{cm}^2 = \frac{1}{2m}\left[m_1 v_1 + m_2 v_2\right]^2$$

$$= \frac{1}{2m}\left[m_1^2 v_1^2 + 2 m_1 m_2 v_1 v_2 + m_2^2 v_2^2\right]$$

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\mu\left[v_1^2 - 2 v_1 v_2 + v_2^2\right]$$

$$= \frac{1}{2m}\left[m_1 m_2 v_1^2 - 2 m_1 m_2 v_1 v_2 + m_1 m_2 v_2^2\right]$$

$$\cancel{T} \quad \frac{1}{2} m v_{cm}^2 + \frac{1}{2} \mu v^2$$

$$= \frac{1}{2m}\left[m_1^2 v_1^2 + m_2^2 v_2^2 + m_1 m_2 v_1^2 + m_1 m_2 v_2^2 \right]$$

$$= \frac{1}{2m}\left[(m_1+m_2) m_1 v_1^2 + (m_1+m_2) m_2 v_2^2 \right]$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = T$$

7.9 $$T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \mu v^2$$

Since $\vec{P}_i = M\vec{V}^{\,i}_{cm} = \vec{P}_f = M\vec{V}^{\,f}_{cm}$

the velocity of the center of mass does not change during the collision.

$$v \equiv v_i = v_{1i} - v_{2i} \qquad v_f = v_{1f} - v_{2f}$$

$$Q = T_i - T_f = \frac{1}{2}\mu v_i^2 - \frac{1}{2}\mu v_f^2$$

$$= \frac{1}{2}\mu v_i^2 \left[1 - \frac{v_f^2}{v_i^2}\right]$$

$$\varepsilon^2 = \frac{v_f^2}{v_i^2}$$

$$Q = \frac{1}{2}\mu v^2 \left[1 - \varepsilon^2\right]$$

7.12

$$\tau = 5.6 \text{ days} = 4.84\times10^{5}\,s$$

$$m_1 = 20\, M_\odot$$

$$m_2 = 16\, M_\odot$$

$$\tau = 2\pi\left[G(m_1+m_2)\right]^{-1/2} a^{3/2}$$

$$\left[\frac{\tau}{2\pi}\sqrt{G(m_1+m_2)}\right]^{2/3} = a$$

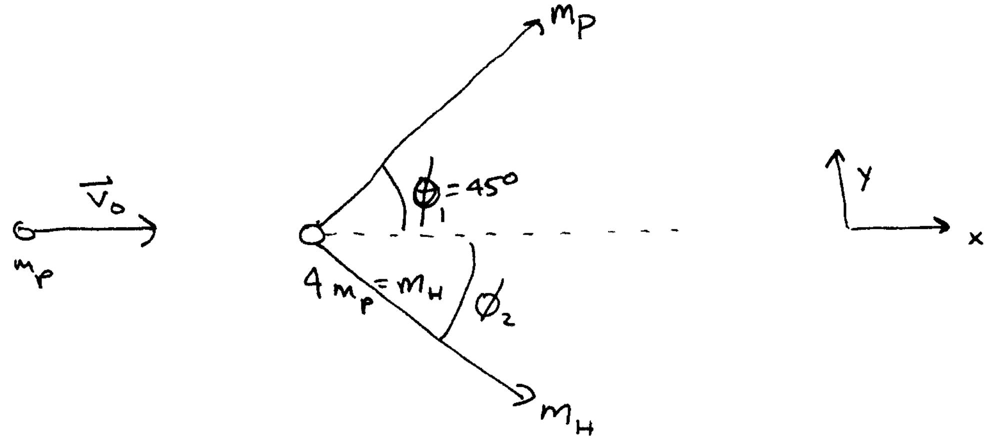
$$a = \left[\frac{4.84\times10^{5}\,s}{2\pi}\left(36\left(6.67\times10^{-11}\frac{Nm^2}{Kg^2}\right)\left(2\times10^{30}kg\right)\right)^{1/2}\right]^{2/3}$$

$$= 3.05\times10^{10}\ m$$

$$1\ AU = 1.5\times10^{11}\ m$$

$$a \approx \frac{1}{5}\ AU$$

7.14



x-component

$$m_p v_0 = m_p v_p^F \cos 45^\circ + 4 m_p v_H^F \cos \phi_2 \qquad (1)$$

y-component

$$0 = m_p v_p^F \sin 45^\circ - 4 m_p v_H^F \sin \phi_2 \qquad (2)$$

Energy

$$\frac{1}{2} m_p v_0^2 = \frac{1}{2} m_p {v_p^F}^2 + \frac{4}{2} m_p {v_H^F}^2$$

$$v_0^2 = {v_p^F}^2 + 4 {v_H^F}^2 \qquad (3)$$

$$(3') \qquad \cancel{\sin\phi_2 =} \qquad v_H^F = \frac{v_p^F \sin 45^\circ}{4 \sin \phi_2}$$

$$v_o - \frac{v_P^F}{\sqrt{2}} = 4 v_H^F \cos\phi_2 \qquad (1)$$

$$\frac{v_P^F}{\sqrt{2}} = 4 v_H^F \sin\phi_2 \qquad (2)$$

$$v_o^2 - \sqrt{2}\, v_P^F v_o + \frac{2 v_P^{F^2}}{2} = 16 v_H^{F^2} \qquad (1)^2 + (2)^2$$

$$4 v_o^2 - 4 v_P^{F^2} = 16 v_H^{F^2} \qquad 4(3)$$

$$-3 v_o^2 - \sqrt{2}\, v_P^F v_o + 5 v_P^{F^2} = 0$$

$$v_P^F = \frac{\sqrt{2}\, v_o \pm \sqrt{2 v_o^2 + 60 v_o^2}}{10}$$

$$= v_o \left[\frac{\sqrt{2}}{10} + \frac{\sqrt{62}}{10} \right]$$

$$= \frac{v_o}{10} \left(\sqrt{2} + \sqrt{62} \right) = 0.9288\, v_o$$

$$\tan\phi_2 = \frac{v_p^F/\sqrt{2}}{v_0 - v_p^F/\sqrt{2}}$$

$$= \frac{1}{\dfrac{\sqrt{2}\,v_0}{v_p^F} - 1} = 1.91$$

$$\phi_2 = 62.41^\circ$$

$$v_H^F = \frac{v_p^F\sqrt{2}}{8\sin 62.41^\circ} = \frac{v_0\,(0.9288)\sqrt{2}}{8\sin 62.41^\circ}$$

$$= 0.185\,v_0$$

$$\vec{V}_H^F = (0.185 v_0 \cos 62\,,\, -0.185 v_0 \sin 62, 0)$$

$$\vec{V}_p^F = (0.9228 v_0 \cos 45\,,\, 0.9228 v_0 \sin 45, 0)$$

7.16

$$\tan \phi_1 = \tan 45^\circ = \frac{\sin\theta}{\gamma + \cos\theta} = 1$$

$$\gamma = \frac{m_1}{m_2} = 1/4$$

$$1 = \frac{\sin\theta}{1/4 + \cos\theta}$$

$$\frac{1}{4} + \cos\theta = \sin\theta$$

$$\sqrt{1-\sin^2\theta} = \sin\theta - 1/4$$

$$1 - \sin^2\theta = \sin^2\theta - \frac{1}{2}\sin\theta + 1/16$$

$$0 = 2\sin^2\theta - \frac{1}{2}\sin\theta - \frac{15}{16}$$

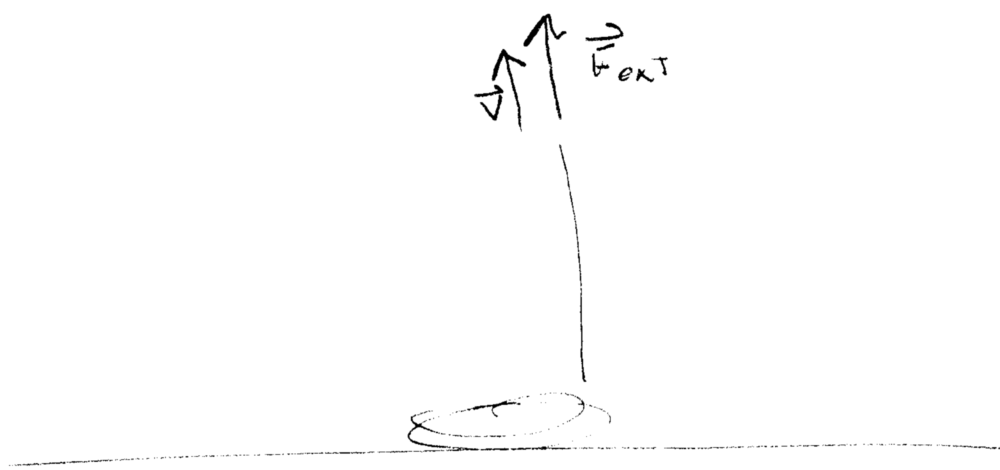
$$\sin^2\theta - \frac{1}{4}\sin\theta - \frac{15}{32} = 0$$

$$\sin\theta = \frac{\frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{15}{8}}}{2}$$

Take + root

$\theta = 55.18^{\circ}$

7.21



Let system be the mass that is lifted off the table.

$$\vec{F}_{ext} + \dot{m}\vec{V}_{add} = m\dot{\vec{V}}_{cm}$$

$$\dot{m} = \gamma v$$ where γ is mass per unit length.

The added mass moves with the velocity of the chain. $\vec{V}_{add} = \vec{v} = -v\hat{k}$, so is moving negatively relative to the chain.

$$\vec{F}_{ext} + \gamma v^2 \hat{k} = m\dot{\vec{V}}_{cm}$$

The location of the center of mass is $z/2$

The velocity of the center of mass is

$$\vec{v}_{cm} = \frac{\dot{z}}{2}\hat{k} = \frac{v_0}{2}\hat{k}$$

$$\dot{\vec{v}}_{cm} = 0$$

$$\vec{F}_{ext} + \gamma v_0^2 \hat{k} = \cancel{\frac{m v_0}{2}\hat{k}} = 0$$

$$\vec{F}_{ext} = -\gamma v_0^2 \hat{k}$$

$$\vec{F}_{pull} \cancel{-} \gamma z g \hat{k} = -\gamma v_0^2 \hat{k}$$

$$\vec{F}_{pull} = \gamma z g - \gamma v_0^2$$

$$= \gamma g \underbrace{\left(z - \frac{v_0^2}{g} \right)}_{\text{length}}$$

7.25

$$\vec{F}_{ext} + \dot{m}\vec{v}_{rel} = m\dot{\vec{v}}_{cm}$$

$$\vec{v}_{rel} = -V\hat{i}$$

$\boxed{\text{Assume rocket moves in } +\hat{i} \text{ direction.}}$

$$\dot{m} = -k$$

$$m(t) = m_0 - kt$$

$$\vec{F}_{ext} = 0$$

$$(-k)(-V\hat{i}) = (m_0 - kt)\dot{\vec{v}}_{cm} = \cancel{(m_0-kt)\dot{v}_{cm}}\,\hat{i}$$

$$= (m_0 - kt)\dot{v}_{cm}\hat{i}$$

$$\frac{kV}{m_0 - kt} = \frac{dv_{cm}}{dt} = \frac{dv_{cm}}{dx_{cm}}\frac{dx_{cm}}{dt}$$

$$= v_{cm}\frac{dv_{cm}}{dx}$$ Nope

$$kV\int_0^t \frac{dt}{m_0 - kt} = \int_0^{v_{cm}} dv = v_{cm}$$

$$u = m_0 - kt \qquad du = -k\,dt$$

$$-V\int_{u_0}^{u}\frac{du}{u} = v_{cm} = -V\ln\frac{u}{u_0} = -V\ln\left(\frac{m_0 - kt}{m_0}\right)$$

$$v_{cm} = -V \ln\left(1 - \frac{kt}{m_0}\right) = \frac{dx}{dt}$$

$$\int_0^d dx = d = -V \int_0^t \ln\left(1 - \frac{kt}{m_0}\right) dt$$

$$u = 1 - \frac{kt}{m_0} \qquad du = -\frac{k}{m_0}\, dt$$

$$d = \frac{V m_0}{k} \int_{u_0}^{u} \ln(u)\, du \qquad u_0 = 1$$

$$\boxed{\int \ln u\, du = u \ln u - u}$$

$$d = \frac{V m_0}{k}\left[u \ln u - u_0 \ln u_0 - (u - u_0)\right]$$

$$= \frac{V m_0}{k}\left[\left(1 - \frac{kt}{m_0}\right)\ln\left(1 - \frac{kt}{m_0}\right) + \frac{kt}{m_0}\right]$$

$$= Vt + \frac{V}{k}\left[(m_0 - kt)\ln\left(1 - \frac{kt}{m_0}\right)\right]$$

The time all fuel is burnt is

$$\varepsilon m_0 = k t_{max}$$

$$t_{max} = \frac{\varepsilon m_0}{k}$$

$$d(t_{max}) = \frac{V \varepsilon m_0}{k} + \frac{V}{k}\left[m_0 - \varepsilon m_0 \ln\left(1-\varepsilon\right)\right]$$

$$= \frac{V m_0}{k}\left[\varepsilon + (1-\varepsilon)\ln(1-\varepsilon)\right]$$

The range increases as more of the rocket becomes fuel.

$$d_{max} = \lim_{\varepsilon \to 1} d(t_{max}) = \frac{V m_0}{k}$$