about Earth. Not only does the perihelion of a satellite's orbit advance, but the plane of the orbit precesses if the satellite is not in Earth's equatorial plane. Detailed analysis of these orbits shows that Earth is basically "pear-shaped and somewhat lumpy."

PROBLEMS

6.1 Find the gravitational attraction between two solid lead spheres of 1 kg mass each if the spheres are almost in contact. Express the answer as a fraction of the weight of either sphere. (The density of lead is 11.35 g/cm^3.)

6.2 Show that the gravitational force on a test particle inside a thin uniform spherical shell is zero

- **(a)** By finding the force directly
- **(b)** By showing that the gravitational potential is constant

6.3 Assuming Earth to be a uniform solid sphere, show that if a straight hole were drilled from pole to pole, a particle dropped into the hole would execute simple harmonic motion. Show also that the period of this oscillation depends only on the density of Earth and is independent of the size. What is the period in hours? ($R_{earth} = 6.4 \times 10^6$ m.)

6.4 Show that the motion is simple harmonic with the same period as the previous problem for a particle sliding in a straight, smooth tube passing obliquely through Earth. (Ignore any effects of rotation.)

6.5 Assuming a circular orbit, show that Kepler's third law follows directly from Newton's second law and his law of gravity: $GMm/r^2 = mv^2/r$.

6.6
- **(a)** Show that the radius for a circular orbit of a synchronous (24-h) Earth satellite is about 7 Earth radii.
- **(b)** The distance to the Moon is about 60 Earth radii. From this calculate the length of the month (period of the Moon's orbital revolution).

6.7 Show that the orbital period for an Earth satellite in a circular orbit just above Earth's surface is the same as the period of oscillation of the particle dropped into a hole drilled through Earth (see Problem 6.3).

6.8 Calculate Earth's velocity of approach toward the Sun, when Earth in its orbit is at an extremum of the latus rectum through the Sun. Take the eccentricity of Earth's orbit to be 1/60 and its semimajor axis to be 93,000,000 miles (see Figure 6.5.1).

6.9 If the solar system were embedded in a uniform dust cloud of density ρ, show that the law of force on a planet a distance r from the center of the Sun would be given by

$$F(r) = -\frac{GMm}{r^2} - \left(\frac{4}{3}\right)\pi\rho m G r$$

6.10 A particle moving in a central field describes the spiral orbit $r = r_0 e^{k\theta}$. Show that the force law is inverse cube and that θ varies logarithmically with t.

6.11 A particle moves in an inverse-cube field of force. Show that, in addition to the exponential spiral orbit of Problem 6.10, two other types of orbit are possible and give their equations.

6.12 The orbit of a particle moving in a central field is a circle passing through the origin, namely $r = r_0 \cos\theta$. Show that the force law is inverse-fifth power.

6.13 A particle moves in a spiral orbit given by $r = a\theta$. If θ increases linearly with t, is the force a central field? If not, determine how θ must vary with t for a central force.

6.14 A particle of unit mass is projected with a velocity v_0 at right angles to the radius vector at a distance a from the origin of a center of attractive force, given by

$$f(r) = -k\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right)$$

If $v_0^2 = 9k/2a^2$,

(a) Find the polar equation of the resulting orbit.

(b) How long does it take the particle to travel through an angle $3\pi/2$? Where is the particle at that time?

(c) What is the velocity of the particle at that time?

6.15 (a) In Example 6.5.4, find the fractional change in the apogee $\delta r_0/r_1$ as a function of a small, fractional change in the ratio of boost speed to circular orbit speed, $\delta(v_0/v_c)/(v_0/v_c)$.

(b) If the speed ratio is 1% too great, by how much would the spacecraft miss the Moon? [This problem illustrates the extreme accuracy needed to achieve a circumlunar orbit.]

6.16 Compute the period of Halley's Comet from the data given in Section 6.5. Find also the comet's speed at perihelion and aphelion.

6.17 A comet is first seen at a distance of d astronomical units from the Sun and it is traveling with a speed of q times the Earth's speed. Show that the orbit of the comet is hyperbolic, parabolic, or elliptic, depending on whether the quantity q^2d is greater than, equal to, or less than 2, respectively.

6.18 A particle moves in an elliptic orbit in an inverse-square force field. Prove that the product of the minimum and maximum speeds is equal to $(2\pi a/\tau)^2$, where a is the semi-major axis and τ is the periodic time.

6.19 At a certain point in its elliptical orbit about the Sun, a planet receives a small tangential impulse so that its velocity changes from v to $v + \delta v$. Find the resultant small changes in a, the semi-major axis.

6.20 (a) Prove that the time average of the potential energy of a planet in an elliptical orbit about the Sun is $-k/a$.

(b) Calculate the time average of the kinetic energy of the planet.

6.21 A satellite is placed into a low-lying orbit by launching it with a two-stage rocket from Cape Canaveral with speed v_0 inclined from the vertical by an elevation angle θ_0. On reaching apogee of the initial orbit, the second stage is ignited, generating a velocity boost Δv_1 that places the payload into a circular orbit (see Figure P6.21).

(a) Calculate the additional speed boost Δv_1 required of the second stage to make the final orbit circular.

(b) Calculate the altitude h of the final orbit. Ignore air resistance and the rotational motion of the Earth. The mass and radius of the Earth are $M_E = 5.98 \times 10^{24}$ kg and $R_E = 6.4 \times 10^3$ km, respectively. Let $v_0 = 6$ km/s and $\theta_0 = 30°$.

6.22 Find the apsidal angle for nearly circular orbits in a central field for which the law of force is

$$f(r) = -k\frac{e^{-br}}{r^2}$$

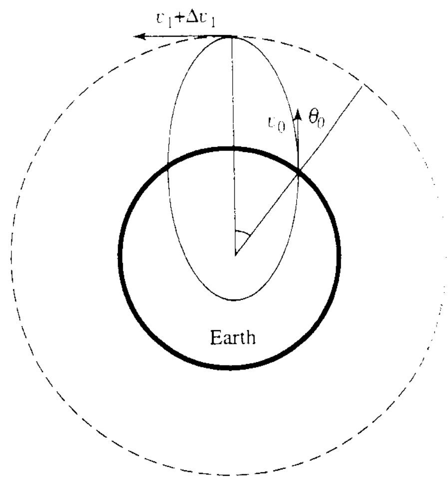


Figure P6.21 Two-stage launch to place satellite in a circular orbit.

6.23 If the solar system were embedded in a uniform dust cloud (see Problem 6.9), what would the apsidal angle of a planet be for motion in a nearly circular orbit? This was once suggested as a possible explanation for the advance of the perihelion of Mercury.

6.24 Show that the stability condition for a circular orbit of radius a is equivalent to the condition that $d^2U/dr^2 > 0$ for $r = a$, where $U(r)$ is the effective potential defined in Section 6.11.

6.25 Find the condition for which circular orbits are stable if the force function is of the form

$$f(r) = -\frac{k}{r^2} - \frac{\epsilon}{r^4}$$

6.26 **(a)** Show that a circular orbit of radius r is stable in Problem 6.22 if r is less than b^{-1}.

(b) Show that circular orbits are unstable in an inverse-cube force field.

6.27 A comet is going in a parabolic orbit lying in the plane of Earth's orbit. Regarding Earth's orbit as circular of radius a, show that the points where the comet intersects Earth's orbit are given by

$$\cos\theta = -1 + \frac{2p}{a}$$

where p is the perihelion distance of the comet defined at $\theta = 0$.

6.28 Use the result of Problem 6.27 to show that the time interval that the comet remains inside Earth's orbit is the fraction

$$\frac{2^{1/2}}{3\pi}\left(\frac{2p}{a} + 1\right)\left(1 - \frac{p}{a}\right)^{1/2}$$

of a year and that the maximum value of this time interval is $2/3\pi$ year, or 77.5 days, corresponding to $p = a/2$. Compute the time interval for Halley's Comet ($p = 0.6a$).

6.29 In advanced texts on potential theory, it is shown that the potential energy of a particle of mass m in the gravitational field of an oblate spheroid, like Earth, is approximately

$$V(r) = -\frac{k}{r}\left(1 + \frac{\epsilon}{r^2}\right)$$

where r refers to distances in the equatorial plane, $k = GMm$ as before, and $\epsilon = 2/5R\,\Delta R$, in which R is the equatorial radius and ΔR is the difference between the equatorial and polar radii. From this, find the apsidal angle for a satellite moving in a nearly circular orbit in the equatorial plane of the Earth, where $R = 4000$ miles and $\Delta R = 13$ miles.

6.30 According to the special theory of relativity, a particle moving in a central field with potential energy $V(r)$ will describe the same orbit that a particle with a potential energy

$$V(r) - \frac{[E - V(r)]^2}{2m_0c^2}$$

would describe according to nonrelativistic mechanics. Here E is the total energy, m_0 is the rest mass of the particle, and c is the speed of light. From this, find the apsidal angle for motion in an inverse-square force field, $V(r) = -k/r$.

6.31 A comet is observed to have a speed v when it is a distance r from the Sun, and its direction of motion makes an angle ϕ with the radius vector from the Sun. Show that the major axis of the elliptical orbit of the comet makes an angle θ with the initial radius vector of the comet given by

$$\theta = \cot^{-1}\left(\tan\phi - \frac{2}{\mathsf{V}^2\mathsf{R}}\csc 2\phi\right)$$

where $\mathsf{V} = v/v_c$ and $\mathsf{R} = r/a_c$ are dimensionless ratios as defined in Example 6.10.1. Apply the result to the numerical values of Example 6.10.1.

COMPUTER PROBLEMS

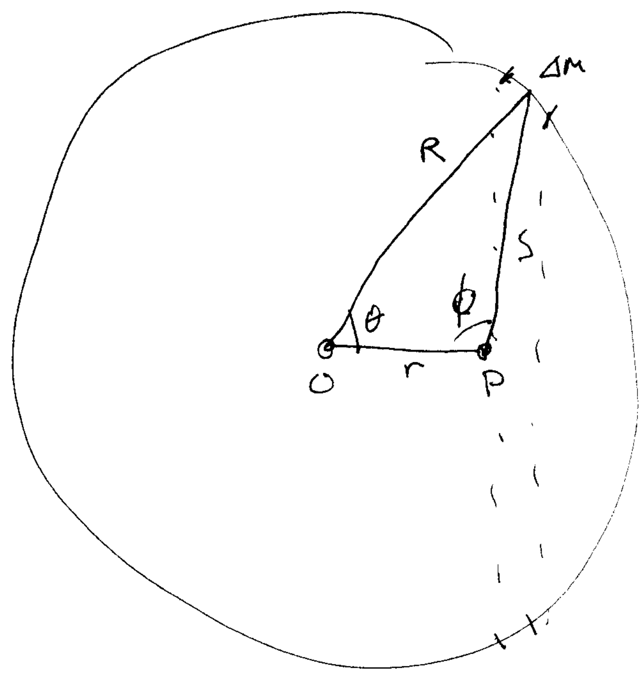
C 6.1 In Example 6.7.2 we calculated the gravitational potential at a point P external to a ring of matter of mass M and radius R. P was in the same plane as the ring and a distance $r > R$ from its center. Assume now that the point P is at a distance $r < R$ from the center of the ring but still in the same plane.

(a) Show that the gravitational potential acting at the point r due to the ring of mass is given by

$$\Phi = -\frac{GM}{R}\left(1 - \frac{r^2}{4R^2} + \cdots\right.$$

Let r = radius of Earth's orbit = 1.496×10^{11} m. R = radius of Jupiter's orbit = 7.784×10^{11} m, and M = mass of Jupiter = 1.90×10^{27} kg. Assume that the average gravitational potential produced by Jupiter on Earth is equivalent to that of a uniform ring of matter around the Sun whose mass is equal to that of Jupiter and whose radius from the Sun is equal to Jupiter's radius.

(6.2)



Law of Cosines

$$s^2 = R^2 + r^2 - 2rR\cos\theta \qquad (1)$$

$$R^2 = s^2 + r^2 - 2sr\cos\phi \qquad (2)$$

Differentiate (1)

$$2s\,ds = 2rR\sin\theta\, d\theta$$

The mass in the dashed strip is -

$$\Delta M = 2\pi r_s \sigma\,(R\,d\theta)$$ σ = surface mass density of shell.

$$= 2\pi R^2 \sin\theta\, \sigma\, d\theta$$

Y-components of the force cancel by symmetry.

$$\Delta F_x = \frac{-m_0 \Delta M G}{s^2} \cos\phi$$

$$= -\frac{m_0 G(2\pi R^2 \sin\theta\, \sigma \Delta\theta)}{s^2} \left[\frac{s^2+r^2-R^2}{2rs}\right]$$

$$R \sin\theta\, d\theta = \frac{s\,ds}{r}$$

$$dF_x = -\frac{2\pi m_0 G R \sigma}{2r^2} \frac{s\,ds}{s^2}\left[\frac{s^2+r^2-R^2}{s}\right]$$

$$= \gamma \frac{ds}{s^2}(s^2+r^2-R^2)$$

$$F_x = \int_{R-r}^{R+r} ds \left[1 + \frac{r^2-R^2}{s^2}\right]$$

$$= 2r - \frac{r^2-R^2}{s}\Bigg|_{R-r}^{R+r}$$

$$= 2r - \frac{r^2-R^2}{R+r} + \frac{r^2-R^2}{R-r}$$

$$= 2r - (r-R) - (r+R) = 0$$

6.7

$$\tau = \sqrt{\frac{4\pi^2}{MG} r^3}$$

$$= \sqrt{\frac{4\pi^2}{MG} R_e^3}$$

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{\sqrt{\frac{4\pi^2}{MG} R_e^3}} = \sqrt{\frac{MG}{R_e^3}}$$

$$= \sqrt{\frac{g}{R_e}}$$

6.8 $\quad \varepsilon = 1/60 \qquad a = 93{,}000{,}000$ miles

Latus rectum $\alpha = (1-\varepsilon^2)a$

The equation of the orbit is

$$r = \frac{\alpha}{1+\varepsilon\cos\theta}$$

$$\dot{r} = \frac{\alpha\varepsilon\sin\theta\,\dot{\theta}}{(1+\varepsilon\cos\theta)^2}$$

At latus rectum, $\theta = \pi/2 \qquad \sin\theta = 1,\ \cos\theta = 0$

$$\dot{r} = \alpha\varepsilon\dot{\theta}$$

Angular Momentum (per mass)

$$l = r^2\dot{\theta}$$

$$\dot{\theta} = \frac{l}{r^2}$$

$$\dot{r} = \frac{\alpha\varepsilon l}{r^2}$$

At lotus rectum, $r = \alpha$

$$\dot{r} = \frac{\varepsilon l}{\alpha}$$

For orbits,

$$\alpha = \frac{m l^2}{k}$$

$$\sqrt{\frac{k\alpha}{m}} = l$$

$$\dot{r} = \frac{\varepsilon}{\alpha}\sqrt{\frac{k\alpha}{m}} = \varepsilon\sqrt{\frac{k}{\alpha m}}$$

$$k = G M_\odot m \qquad \alpha = (1-\varepsilon^2) a$$

$$\dot{r} = \varepsilon\sqrt{\frac{G M_\odot}{(1-\varepsilon^2) a}}$$

$$M_\odot = 333{,}480\, M_e = (333{,}480) \times 5.98\times 10^{24}\,\text{kg}$$

$$= 2 \times 10^{30}\,\text{kg}$$

$$\alpha = (1-\varepsilon^2) a \approx a$$

$$a = 1.495 \times 10^{11}\,\text{m}$$

$$\dot{r} = \frac{1}{60}\sqrt{\frac{\left(6.67\times10^{-11}\,\frac{N\,m^2}{kg^2}\right)\left(2\times10^{30}\,kg\right)}{1.495\times10^{11}\,m}}$$

$$= 498\ m/s$$

6.9 From the oscillation through the earth,

$$\vec{g} = -\frac{\rho r}{3\varepsilon_g^o}\hat{r}$$

from a uniform volume mass.

$$\varepsilon_g^o = \frac{1}{4\pi G}$$

$$\frac{MG}{r^2} = \frac{M}{4\pi\varepsilon_0^g r^2}$$

$$G = \frac{1}{4\pi\varepsilon_0^g}$$

$$\vec{g} = -\frac{4\pi}{3}G\rho r$$

$$F(r) = \frac{-mMG}{r^2} \quad -\frac{4}{3}\pi G\rho r$$

6.11

$$f = -\frac{\alpha}{r^3} = -\alpha u^3$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2} f(u^{-1}) = \frac{\alpha u}{ml^2}$$

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\alpha}{ml^2}\right) u = 0$$

Three Cases

(I) $1 - \frac{\alpha}{ml^2} = 0$ $C + Dt$

(II) $1 - \frac{\alpha}{ml^2} > 0$ oscillation

(III) $1 - \frac{\alpha}{ml^2} < 0$ exponential decay.

6.12

$$r = r_0 \cos\theta$$

$$u = \frac{1}{r} = \frac{1}{r_0 \cos\theta}$$

$$\frac{du}{d\theta} = \frac{1}{r_0}\left[\frac{\sin\theta}{\cos^2\theta}\right] = \frac{1}{r_0} \sin\theta \cos^{-2}\theta$$

$$\frac{d^2u}{d\theta^2} = \frac{1}{r_0}\left[\frac{1}{\cos\theta} + 2\frac{\sin^2\theta}{\cos^3\theta}\right]$$

$$= \frac{1}{r_0 \cos\theta}\left[1 + 2\frac{(1-\cos^2\theta)}{\cos^2\theta}\right]$$

$$= \frac{1}{r_0 \cos\theta}\left[1 + \frac{2}{\cos^2\theta} - 2\right]$$

$$= \frac{1}{r_0 \cos\theta}\left[\frac{2}{\cos^2\theta} - 1\right]$$

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{r_0 \cos\theta}\left[\frac{2}{\cos^2\theta} - 1\right] + \frac{1}{r_0 \cos\theta}$$

$$= \frac{2}{r_0 \cos^3\theta} = \frac{-1}{m l^2 u^2} f(u^{-1})$$

$$f(u^{-1}) = -\frac{2 m l^2 u^2}{r_0 \cos^3\theta} = \frac{-2 m l^2}{r_0^3 \cos^5\theta}$$

$$= \frac{-2 m l^2 r_0^2}{r_0^5 \cos^5\theta}$$

$$f(r) = -\frac{2ml^2 r_0^2}{r^5}$$

6.14

$$f(r) = -k\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right)$$

$$f(u^{-1}) = -k\left(4u^3 + a^2 u^5\right)$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{m\ell^2 u^2} f(u^{-1})$$

$$= \frac{k}{m\ell^2}\left(4u + a^2 u^3\right)$$

Stuck

$$V(r) = -\int f(r)\,dr = -k\left(\frac{2}{r^2} + \frac{a^2}{4r^4}\right)$$

<u>Total Energy Conserved</u> $r(0) = a$ $\vec{v}(0) = \sqrt{\frac{9k}{2a^2m}}\,\hat{e}_\theta$

$$E = T_0 + V_0 = \frac{1}{2}mv_0^2 + V(a)$$

$$= \frac{9k}{4a^2} - k\left(\frac{2}{a^2} + \frac{1}{4a^2}\right)$$

$$= \frac{k}{a^2}\left(\frac{9}{4} - \frac{1}{4} - 2\right) = 0$$

$$0 = \frac{1}{2}mv^2 + V(r)$$

6.14(b)

$$\vec{v}^2 = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$0 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r^2}\left(2 + \frac{a^2}{4r^2}\right)$$

<u>Conserve Angular Momentum</u> $\quad l = r^2\dot{\theta}$

$$\dot{\theta} = \frac{l}{r^2}$$

$$0 = \dot{r}^2 + \frac{l^2}{r^2} - \frac{2k}{mr^2}\left(2 + \frac{a^2}{4r^2}\right)$$

$$0 = \dot{r}^2 + \left(l^2 - \frac{4k}{m}\right)\frac{1}{r^2} - \frac{2ka^2}{4mr^4}$$

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$$

$$0 = \left(\frac{dr}{d\theta}\right)^2 \frac{l^2}{r^4} + \left(l^2 - \frac{4k}{m}\right)\frac{1}{r^2} - \frac{2ka^2}{4mr^4}$$

$$l = \cancel{m}\,a v_0 \qquad l^2 = a^2 v_0^2 = \frac{9k}{2m}$$

(b) $\theta = \frac{3\pi}{2}$

$$r(\theta) = a\cos\frac{\theta}{3} = a\cos\frac{\pi}{2} = 0$$

$$l = r^2\dot{\theta}$$

$$\dot{\theta} = \frac{l}{r^2} = \frac{l}{a^2\cos^2\theta/3}$$

$$\int_0^{\theta} \cos^2\frac{\theta}{3}\, d\theta = \frac{l}{a^2} t$$

$$\cancel{3\sin\theta/3 = \frac{l}{a^2}t}$$

$$\int \cos^2 au\, du = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\cancel{3\sin\pi/2 = \frac{l}{a^2}t}$$

$$\cancel{t = \frac{3a^2}{l} =}$$

$$\frac{\theta}{2} + \frac{3\sin\frac{2}{3}\theta}{4} = \frac{l}{a^2}t$$

$$\frac{a^2}{l}\frac{3\pi}{4} = t \qquad \frac{a^2}{av_0}\frac{3\pi}{4} = t$$

$$l = av_0 \qquad \left(\frac{a}{v_0}\right)\frac{3\pi}{4}$$

(c) $r\left(\frac{3\pi}{2}\right) = 0$

$J = r^2 \dot{\theta} = \text{const}$

$\dot{\theta}\left(\frac{3\pi}{2}\right) = \infty$

6.17 From example 6.10.1

$$\varepsilon = \left[1 + \left(v^2 - \frac{2GM_\odot}{r}\right)\left(\frac{rv\sin\phi}{GM_\odot}\right)\right]^{1/2}$$

$\varepsilon < 1$ ellipse

$\varepsilon = 1$ parabolic

$\varepsilon > 1$ hyperbolic

Depends on sign of $v^2 - \frac{2GM_\odot}{r}$ < 1 ellipse

$= 1$ parabola

> 1 hyperbolic

In astronomical units $GM_\odot = 1$

Use normalized version 6.10.1,

$$\varepsilon = \left[1 + \left(V^2 - \frac{2}{R}\right)(RV\sin\phi)^2\right]^{1/2}$$

$V = q$, $R = d$

$$\left(V^2 - \frac{2}{R}\right) = 0 \implies \varepsilon = 1$$

$q^2 d = 2$ is transition point.

(6.19) At θ a planet recieves a tangental impulse, changing $v \to v + \delta v$

$\Rightarrow$ Changes kinetic energy but not potential

$$E = \frac{1}{2} m v^2 + V(r)$$

$$\delta E = m v \, \delta v$$

How does total energy depend on semi-major axis?

$$E = -\frac{k}{2a} \qquad (6.10.10)$$

$$\delta E = +\frac{k}{2}\left(\frac{\delta a}{a^2}\right)$$

$$\frac{\delta a}{a^2} \frac{k}{2} = m v \, \delta v$$

Can be simplified further if we assume circular orbit.