Section 4 - Equations of Motion under a central Force

EOM

$$\vec{F} = f(r)\hat{e}_r = m\vec{a}$$

$$= m\left([\ddot{r} - r\dot{\theta}^2]\hat{e}_r + [2\dot{r}\dot{\theta} + r\ddot{\theta}]\hat{e}_\theta \right)$$

Component EOM

$$f(r) = m[\ddot{r} - r\dot{\theta}^2] \qquad (\hat{e}_r)$$

$$0 = m(2\dot{r}\dot{\theta} + r\ddot{\theta}) \qquad (\hat{e}_\theta)$$

Look at $\hat{e}_\theta$

$$l = r^2\dot{\theta} = \text{constant}$$

$$\frac{dl}{dt} = 2\dot{r}\dot{\theta}r + r^2\ddot{\theta} = (2\dot{r}\dot{\theta} + r\ddot{\theta})r$$

$$= 0$$

So the $\hat{e}_\theta$ is a restatement of the conservation of angular momentum.

Solve $$f(r) = m\left[\ddot{r} - r\dot{\theta}^2\right]$$

$$r = \frac{1}{u} \qquad \dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -l\frac{du}{d\theta}$$

since $l = u^2\dot{\theta}$

$$\ddot{r} = -l\frac{d}{dt}\frac{du}{d\theta}$$

$$= -l\dot{\theta}\frac{d^2u}{d\theta^2} = -l^2u^2\frac{d^2u}{d\theta^2}$$

$$f(u^{-1}) = m\left[-l^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}\left(l^2u^4\right)\right]$$

Differential Eqn of Orbit

$$-\frac{f(u^{-1})}{ml^2u^2} = \frac{d^2u}{d\theta^2} + u$$

Example $r = \theta$ $\qquad u = \frac{1}{r} = \frac{1}{\theta}$

$$\frac{du}{d\theta} = -\frac{1}{\theta^2} = -u^2 \qquad \frac{d^2u}{d\theta^2} = 2\frac{1}{\theta^3} = 2u^3$$

$$\frac{-f(u^{-1})}{m l^2 u^2} = 2u^3 + u$$

$$f(u^{-1}) = -m l^2 \left[2u^5 + u^3 \right]$$

$$f(r) = -m l^2 \left[\frac{2}{r^5} + \frac{1}{r^3} \right]$$

Lecture 3/5/2003

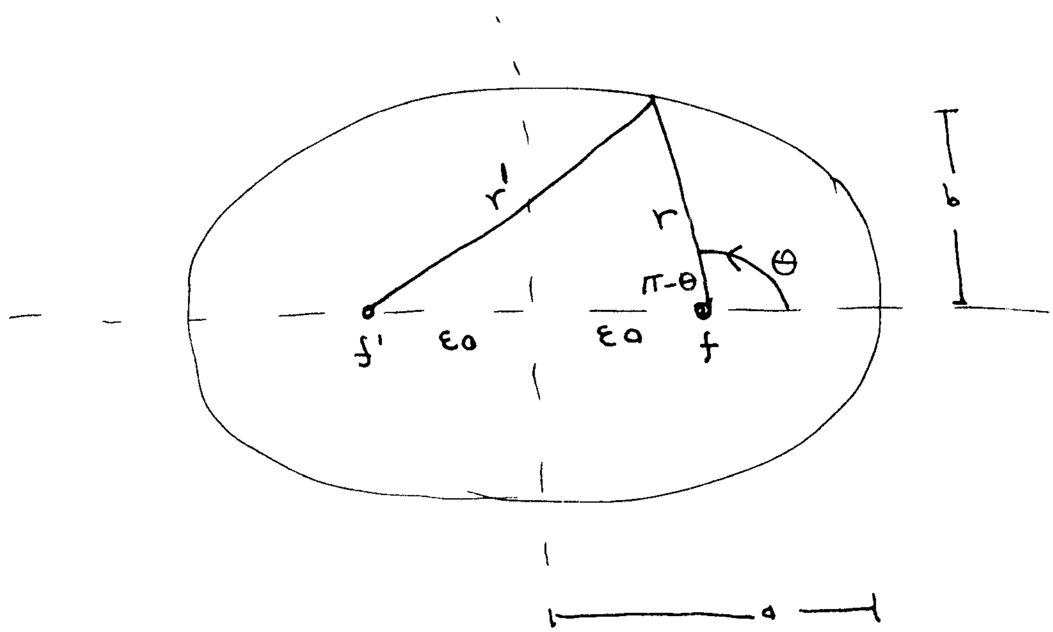
Inverse Square Forces.

- Test March 10 6:00 pm in Chem 113

Section Ellipses

Ellipse – Locus of points whose distance is a total $2a$ from two points (Foci)

Eccentricity (ε) - Fraction of the length of major axis a the foci displaced from the center



$\Rightarrow$ If $\varepsilon = 0$, circle.

Build Equation Around one Focus

By definition $r + r' = 2a$ $\qquad r' = r - 2a$

$\Rightarrow r'^2 = r^2 - 4ra + 4a^2$

Law of Cosines

$$r'^2 = r^2 + (2\varepsilon a)^2 - 2r(2\varepsilon a)\cos(\pi - \theta)$$

$$\cos(\pi - \theta) = -\cos\theta$$

$$r^2 - 4ra + 4a^2 = r^2 + 4\varepsilon^2 a^2 + 4\varepsilon r a \cos\theta$$

$$4a^2(1-\varepsilon^2) = 4ra(1+\varepsilon\cos\theta)$$

$$\frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta} = \boxed{r = \frac{\alpha}{1+\varepsilon\cos\theta}}$$

Lotus rectum $\alpha = a(1-\varepsilon^2)$ Perpendiclar distance to ellipse from foci.

Pericenter (Closest approach) $r_0 = \dfrac{\alpha}{1+\varepsilon} = a(1-\varepsilon)$

Apocenter (Farthest) $r_1 = \dfrac{\alpha}{1-\varepsilon} = a(1+\varepsilon)$

Section Inverse Square Forces

$$\vec{F} = \frac{k q_1 q_2}{r^2} \hat{e}_r \quad \text{or} \quad \vec{F} = -\frac{G m_1 m_2}{r^2} \hat{e}_r$$

$$\vec{F} = f(r)\, \hat{e}_r \qquad f(k) = -\frac{k}{r^2}$$ ← not EM k

$$k = G m_1 m_2$$

<u>Use equation of orbit</u>

$$\frac{d^2 u}{d\theta^2} + u = -\frac{1}{m l^2 u^2} f(u^{-1})$$

$$f(u^{-1}) = -k u^2$$

$$\frac{d^2 \theta}{d\theta^2} + u = \frac{k}{m l^2} \qquad \underline{\text{SHO}}$$

$$\frac{1}{r} = u = A \cos(\theta - \theta_0) + \frac{k}{m l^2} \qquad (\omega_0^2 = 1)$$

A, θ_0 constants of integration

$$r = \frac{1}{\frac{k}{m l^2} + A \cos(\theta - \theta_0)} = \frac{\frac{m l^2}{k}}{1 + \frac{A m l^2}{k} \cos(\theta - \theta_0)}$$

Let $\theta_0 = 0$ at $t = 0$.

$$\alpha = \frac{m l^2}{K}$$

$$\varepsilon = \frac{A m l^2}{K}$$

At pericenter of apocenter, $l = v_0 r_0$ or $l = v_1 r_1$

From Energy Equation of Orbit

$$\varepsilon = \sqrt{1 + \frac{2 \alpha E}{K}} \qquad \text{(Elliptical orbits)}$$

and

$$\boxed{E = \frac{-k}{2a}} = \frac{1}{2} m v^2 - \frac{K}{r}$$

If $E > 0$ hyperbolic orbit

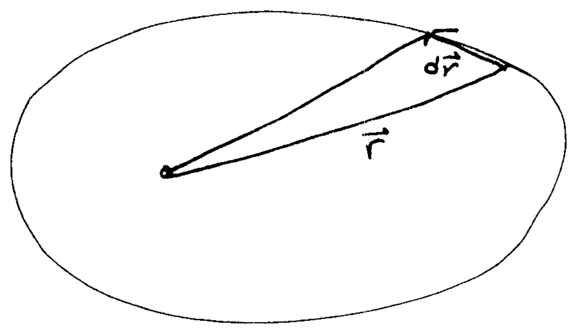
$E = 0$ parabola

$E < 0$ elliptical

Section Kepler's Laws

I. Law of Ellipses – Planets travel in elliptical orbits.

II Law of Equal Areas – A line drawn between planet and sun sweeps out equal areas in equal times.



Parallelogram

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| = \frac{1}{2}|\vec{r} \times \vec{v}|dt$$

$$= \frac{1}{2} \ell \, dt$$

$$\dot{A} = \frac{\ell}{2} = \text{constant}$$

II <u>Harmonic Law</u>

Period $\displaystyle\int_0^A dA = \frac{\ell}{2}\int_0^\tau dt$

$$A = \frac{\ell}{2}\tau$$

Area of Ellipse $A = \pi a b$

$$= \pi a^2 (1-\varepsilon^2)^{1/2}$$

$$\tau = \frac{2\pi a^2}{l} (1-\varepsilon^2)^{1/2}$$

$$\alpha = \frac{m}{k} l^2$$

$$l = \sqrt{\frac{k\alpha}{m}} = \sqrt{\frac{k}{m}} \sqrt{a} \sqrt{1-\varepsilon^2}$$

$$\tau = \frac{2\pi a^{3/2}}{\sqrt{k/m}}$$

$$\tau^2 = \frac{4\pi^2 m}{k} a^3$$

If τ measured is Earth years and a measured in AU, $\tau^2 = a^3$

1 AU = 1.5×10^{11} m = 1 earth radius.

Lecture 3/7

Examples of Circular Motion

Test Monday Night March 10 6:00pm
in Chem 113.

Section - Let's Work on Earth

$\tau = 1$ yr.

$r_0 =$ Perihelion Jan 2 Distance 147.09×10^9 m] NASA

$r_1 =$ Aphelion July 2 Distance 152.10×10^9 m]

$M_\odot = 2 \times 10^{30}$ kg

$G = 6.673 \times 10^{-11} \frac{Nm^2}{kg^2}$] NIST

$$r_0 = \frac{\alpha}{1+\varepsilon} \qquad r_1 = \frac{\alpha}{1-\varepsilon}$$

$$r_0(1+\varepsilon) = \alpha = r_1(1-\varepsilon)$$

$$\frac{r_1 - r_0}{r_1 + r_0} = \varepsilon = \frac{152.10 - 147.09}{152.10 + 147.09} = 0.0167$$

What's α?

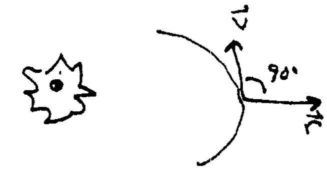
$$\alpha = r_0(1+\varepsilon) = 149.55 \times 10^9 \text{ m}$$

$$= \frac{m l^2}{k}$$

$$l = \left(\frac{k}{m}\alpha\right)^{1/2} = (M_\odot G \alpha)^{1/2}$$

$$= 4.47 \times 10^{15} \text{ m}^2/\text{s}$$

Velocity at Perihelion



$$l = |\vec{r} \times \vec{v}| = r_0 v_0$$

$$v_0 = \frac{l}{r_0} = 30.4 \text{ km/s}$$

Velocity at Aphelion

$$v_1 = \frac{l}{r_1} = 29.4 \text{ km/s}$$

Period

$$\tau = \left(\frac{4\pi^2}{G M_\odot} a^3 \right)^{1/2}$$

Semi-Major Axis

$$(1 - \varepsilon) a = r_0$$

$$a = \frac{r_0}{1-\varepsilon} = \frac{r_1}{1+\varepsilon}$$

= ~~142.04 ×10⁹ m~~ $= 149.59 \times 10^9$ m

149.60×10^9 m

Period

$$\tau = 2\pi \left(\frac{a^3}{GM_\odot}\right)^{1/2}$$

$$= 3.147 \times 10^7 \, s$$

$$1 yr = 365 \times 24 \times 60 \times 60 = 3.15 \times 10^7 s$$

> I was asked if $\frac{v^2}{r}$ was still the centripetal acceleration, it always is. In circular orbit $\frac{v^2}{r}$ exactly balances central force. In elliptical orbit, the planet accelerates toward or away from sun

Velocity Now

$$r = \frac{a}{1 + \varepsilon \cos\theta}$$

$$\dot{r} = \frac{a \varepsilon \sin\theta \, \dot{\theta}}{(1 + \varepsilon \cos\theta)^2}$$

$$\dot{\theta} = \frac{l}{r^2} = \frac{l(1 + \varepsilon \cos\theta)^2}{a^2}$$

$$\boxed{\dot{r} = \frac{a \varepsilon l}{a^2} \sin\theta}$$

Progress but what is θ now

$$\dot{\theta} = \frac{l}{r^2} = (1 + \varepsilon \cos\theta)^2 \frac{l}{a^2} = \frac{d\theta}{dt}$$

$$\frac{l}{\alpha^2}\,dt = \frac{l}{\alpha^2}t = \int_0^\theta \frac{d\theta}{(1+\varepsilon\cos\theta)^2}$$

$$= \text{Nothing Nice}$$

<u>Approximate Orbit as circular</u>

March 5 - ~~Jan~~ Feb 2

30 + 28 + 5 = 63 days

$$\theta = 360° \cdot \frac{63}{365} =$$

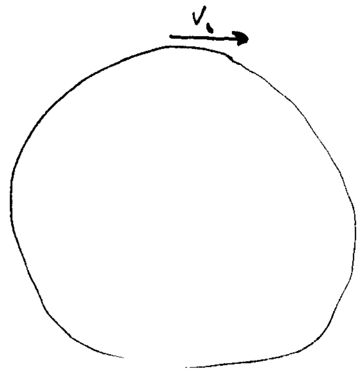
$$\theta = 62°$$

$$\dot{r} = \frac{\varepsilon l}{\alpha} \sin 62 = \frac{(0.0167)(4.47\times10^{15}\,m^2/s)}{149.55\times10^9\,m} \sin 62$$

$$= 441\ m/s$$

Section - Tossing Coin Near Earth

Question - I toss a coin to Justin, how fast and in what direction for circular orbit?



Answer Parallel to ground. Must be a piece of orbit.

$$\frac{mv^2}{r} = +\frac{k}{r^2} \quad (\text{Downward } +)$$

$$v = \sqrt{\frac{k}{mr}} = \sqrt{\frac{M_\oplus G}{r_e}}$$

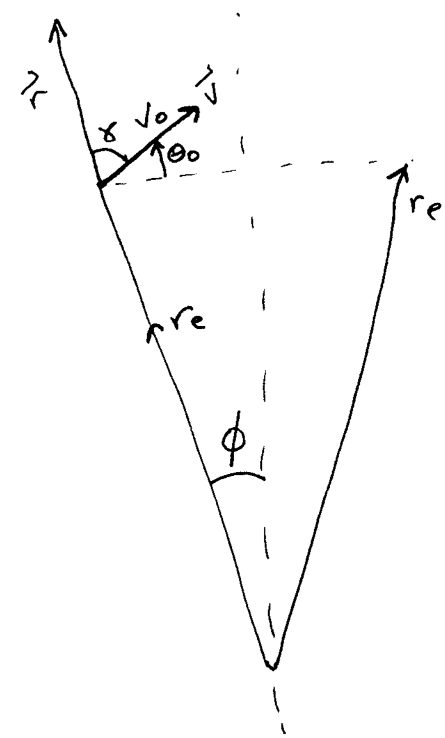
$$= \sqrt{\frac{M_\oplus G}{r_e^2} r_e} = \sqrt{g r_e}$$

$$= 7924 \text{ m/s} \qquad \frac{M_\oplus G}{r_e^2} = g$$

Question - Now I lob the coin to Justin at initial velocity $v_0 = 5\,m/s$ and angle $\theta_0 = 45^\circ$ (to ground). The earth collapses into black hole (size of small marble). What is the orbit of Justin? What is the orbit of the coin if Justin misses?

Answer Justin $\ell = 0$, falls straight down.

Coin



$\ell = |\vec{r} \times \vec{v}| = r_e v_0 \sin\gamma$

$\phi \ll 1$

$\gamma \approx \pi/2 - \theta_0$

$\ell = r_e v_0 \sin(\pi/2 - \theta_0)$

$\ell = r_e v_0 \cos\theta_0$

$$\ell = (6.4 \times 10^6\,m)(5\,m/s)\cos 45^\circ = 2.26 \times 10^7\,m^2/s$$

$$\alpha = \frac{m\ell^2}{k} = \frac{\ell^2}{M_\otimes G} = \frac{\ell^2}{g r_e^2} \qquad M_\otimes G = g r_e^2$$

$$\alpha = 1.27\,m$$

<u>Find Semi-Major Axis</u>

$$E = -\frac{k}{2a} = \frac{1}{2} m v_0^2 - \frac{k}{r_e}$$

$$a = \frac{-k/2}{\frac{1}{2} m v_0^2 - k/r_e} = \frac{r_e}{2 - \frac{m v_0^2 r_e}{k}}$$

$$\frac{m r_e}{k} = \frac{r_e}{M_\otimes G} = \frac{1}{g r_e}$$

$$a = \frac{r_e}{2 - \frac{v_0^2}{g r_e}} = \frac{r_e}{2 - 4\times10^{-7}} \approx r_e/2$$

<u>Find Eccentricity</u>

$$r_1 \approx r_e = \frac{\alpha}{1-\varepsilon} \Rightarrow r_e(1-\varepsilon) = \alpha$$

$$\varepsilon = 1 - \frac{\alpha}{r_e} = 0.99999998016$$

How close at perigee,

$$r_0 = \frac{a}{1+\varepsilon} = 0.635 \, m$$