Section - Work and Force

We need to find a way to put forces into Lagrange's equations and get the forces of constraint out.

Conservative Forces - Generalized Force Q_i

$$Q_i = -\frac{\partial V}{\partial q_i}$$

Unfortunately, these forces are already included in the Lagrangian in the potential function. Suppose we wanted to write part of the Lagrangian in terms of these generalized forces.

$$L = T - (V + V') = L' - V'$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L'}{\partial q_i} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial V'}{\partial q_i} = 0$$

$$\frac{\partial L'}{\partial q_i} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} + Q_i = 0$$

How do we find generalized force in general?

Virtual Displacement (δq_i) - Any infinitesimal displacement of system

Virtual Work δW - Work done by the virtual displacement

$$Q = \frac{\delta W}{\delta q_i}$$ (work is force × distance)

Relate virtual displacements to real forces

$$Q = \sum_i F_i \frac{\partial x_i}{\partial q_i}$$

where equations $x(q_1 \cdots q_n, t)$ define generalized coordinates.

Section - Forces of Constraint

So we can add forces to Lagrange's equations, how do we extract forces of constraint.

Let $\lambda_1(t)$ be the force associated with constraint equation $f_1(q_1 \cdots q_n, t)$. The force will be directed $\perp$ to the surface defined by the constraint.

The virtual work δW to deform the system δq_1, possibly not in conformance with the constraint is

$$\delta W = \lambda_1 \times \text{ distance move } \perp \text{ surface.}$$

$$= \lambda_1 \frac{\partial f_1}{\partial q_i} \delta q_i$$

$$Q_i\,(\text{constraint}) = \lambda_1 \frac{\partial f_1}{\partial q_i}$$

~~where λ_1 is magnitude of constraint force.~~

The full constraint force vector is

$$\vec{R} = \lambda_1 \left(\frac{\partial f}{\partial q_1}, \frac{\partial f}{\partial q_2}, \cdots \right)$$

If it turns out $\vec{Q} = \lambda(1, 0, 0, 0)$ then λ is the magnitude of the constraint force.

We have been using our constraint equations to lower the degrees of freedom of the system of equations. We could also include the force of constraint and solve the full $3N$ equations

$\Rightarrow$ Harder

$\Rightarrow$ Find λ_i

The λ_i are called Lagrange multipliers and if we have two generalized coordinates connected by an equation of constraint $f(q_1, q_2, t) = 0$

EOM

$$\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} + \lambda \frac{\partial f}{\partial q_1} = 0$$

$$\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} + \lambda \frac{\partial f}{\partial q_2} = 0$$

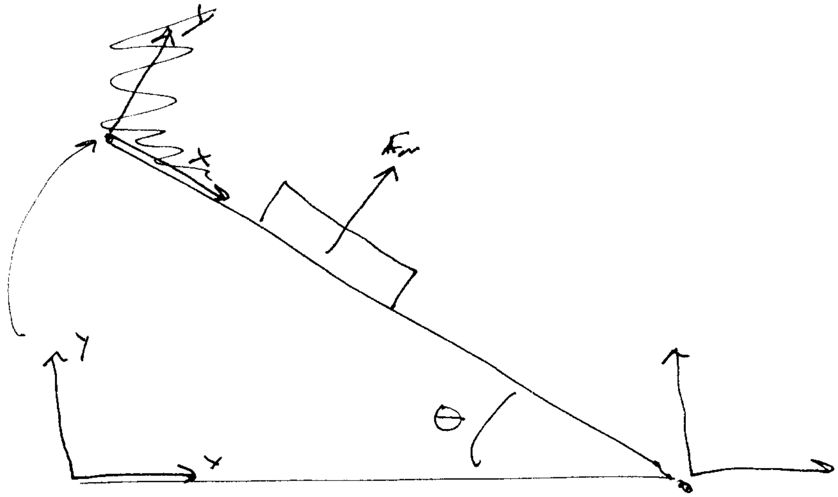
$$f(q_1, q_2, t) =$$

3 eqns

3 unknowns

q_1, q_2, λ

Example — Mass sliding down frictionless plane



$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$V = mgy$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

x - eqn

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda \frac{\partial f}{\partial x} = 0$$

$$y = -\tan\theta \, x$$

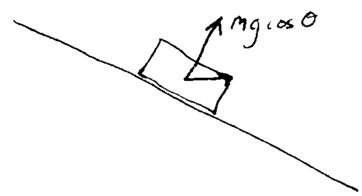
$$f = y + \tan\theta \, x = 0$$

$$\frac{\partial f}{\partial x} = \tan\theta \qquad \frac{\partial f}{\partial y} = 1$$

Our force of constraint will be

$$\vec{Q} = \lambda\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \lambda\left(\tan\theta, 1\right) = \vec{F}_N$$

The magnitude of $|\vec{F}_N| = mg\cos\theta$



$$\vec{Q} = \left(mg\cos\theta\sin\theta,\ mg\cos^2\theta\right)$$

$$\lambda = mg\cos^2\theta$$

<u>x-eqn</u> $\quad -m\ddot{x} + \lambda\tan\theta = 0 \quad \Rightarrow \quad m\ddot{x} = \lambda\tan\theta$

<u>y-eqn</u> $\quad -mg - m\ddot{y} + \lambda = 0$

<u>constraint</u> $\quad \ddot{y} = -\tan\theta\,\ddot{x}$

$$-mg + m\ddot{x}\tan\theta + \lambda = 0$$

$$-mg + \lambda(1+\tan^2\theta) = 0$$

$$\lambda = \frac{mg}{1+\tan^2\theta} = mg\cos^2\theta$$

Section - Euler Angles

The direction cosines with respect to the fixed axes are not a set of independent angles to describe the rotation of a three-dimensional rigid body.

$\Rightarrow$ Try rotating solid, ~~you will find~~

Independent Rotations - Euler Angles

ϕ - Rotation about z axis, $x \to x'$, $y \to y'$, $z \to z$

θ - Rotation about x' axis, $z \to z'$, $y' \to y''$, $x' \to x'$

ψ - Rotation about z' axis, $z' \to z'$, $y'' \to y'''$, $x' \to x''$

Look at double gimble mount picture

Example 8.14. *Equations of motion of a top.*

Imagine the body, Fig. 8-16, replaced by a top with the tip stationary at O and its axis of symmetry along Z. Take X, Y, Z axes shown as body-fixed.

Since $I_x = I_y$ and O is at rest, T simplifies and L may be written as

$$L \quad = \quad \tfrac{1}{2}[I_x(\dot\theta^2 + \dot\psi^2 \sin^2\theta) \;+\; I_z(\dot\phi + \dot\psi\cos\theta)^2] \;-\; Mgr\cos\theta$$

where M is the total mass and r the distance along the axis of symmetry from O to c.m. Applying Lagrange's equation, the following equations of motion are obtained.

$$\begin{aligned} I_x\ddot\theta + [(I_z - I_x)\dot\psi\cos\theta + I_z\dot\phi]\dot\psi\sin\theta &= Mgr\sin\theta \\ I_z(\dot\psi\cos\theta + \dot\phi) \;=\; P_\phi &= \text{constant} \\ I_x\dot\psi\sin^2\theta + P_\phi\cos\theta \;=\; P_\psi &= \text{constant} \end{aligned} \qquad (8.13)$$

Detailed treatments of these equations, which may be found in many books, will not be repeated here. See, for example: *Gyrodynamics and its Engineering Applications* by R. N. Arnold and L. Maunder, Chapter 7, Academic Press, 1961; or *A Treatise on Gyrostatics and Rotational Motion* by Andrew Gray, Chapter V, The Macmillan Co., 1918. The latter book gives extensive treatments of tops, gyroscopes, etc.

Example 8.15. *Kinetic energy of top with tip free to slide on the smooth X_1Y_1 plane.*

Assuming body-fixed axes as in Example 8.14 and locating O (the tip) by (x_1, y_1), the first term of *(8.10)* is merely $\tfrac{1}{2}M(\dot x_1^2 + \dot y_1^2)$. Expressions for $\omega_x, \omega_y, \omega_z$ are as before. Hence the second term is $\tfrac{1}{2}[I_x(\dot\theta^2 + \dot\psi^2\sin^2\theta) \;+\; I_z(\dot\phi + \dot\psi\cos\theta)^2]$. Since $\bar x = \bar y = 0$, $\bar z = r$, the third term of T reduces to $Mr[v_{0x}\omega_y - v_{0y}\omega_x]$ in which v_{0x}, v_{0y} must be components of $\mathbf{v}_0$ along the instantaneous directions of X, Y respectively. That is, $v_{0x} = \dot x_1\alpha_{11} + \dot y_1\alpha_{12}$, $v_{0y} = \dot x_1\alpha_{21} + \dot y_1\alpha_{22}$. Note that v_{0z} is not required. Introducing these and expressions for ω_y, ω_x completes the third term.

Equations of motion corresponding to $x_1, y_1, \theta, \psi, \phi$ can be obtained at once. Note that, assuming the X_1Y_1 plane smooth, the reactive force on the tip will not appear in the generalized forces.

An alternative method of obtaining T, requiring perhaps less tedious work, is the following. Imagine body-fixed axes taken as above but with origin at c.m. Since $\bar x = \bar y = \bar z = 0$, the third term of *(8.10)* drops out. $\mathbf{v}_0$, not so simple as before, may be obtained from the following relations. Coordinates x, y, z of the origin of body-fixed axes relative to the X_1, Y_1, Z_1 frame are $x = x_1 + r\alpha_{31}$, $y = y_1 + r\alpha_{32}$, $z = r\alpha_{33}$. Differentiating and substituting into $v_0^2 = \dot x^2 + \dot y^2 + \dot z^2$, we have an appropriate expression for v_0^2. Hence

$$T \quad = \quad \tfrac{1}{2}Mv_0^2 \;+\; \tfrac{1}{2}[\bar I_x^p(\omega_x^2 + \omega_y^2) + \bar I_z^p\omega_z^2]$$

where $\omega_x, \omega_y, \omega_z$ are the same as in the first part of the example and $\bar I_x^p, \bar I_z^p$ are principal moments of inertia through c.m. (As a third method we can write $v_0^2 = \dot x^2 + \dot y^2 + r^2\dot\theta^2\sin^2\theta$.)

Example 8.16. *Kinetic energy and equations of motion of the gyroscope.*

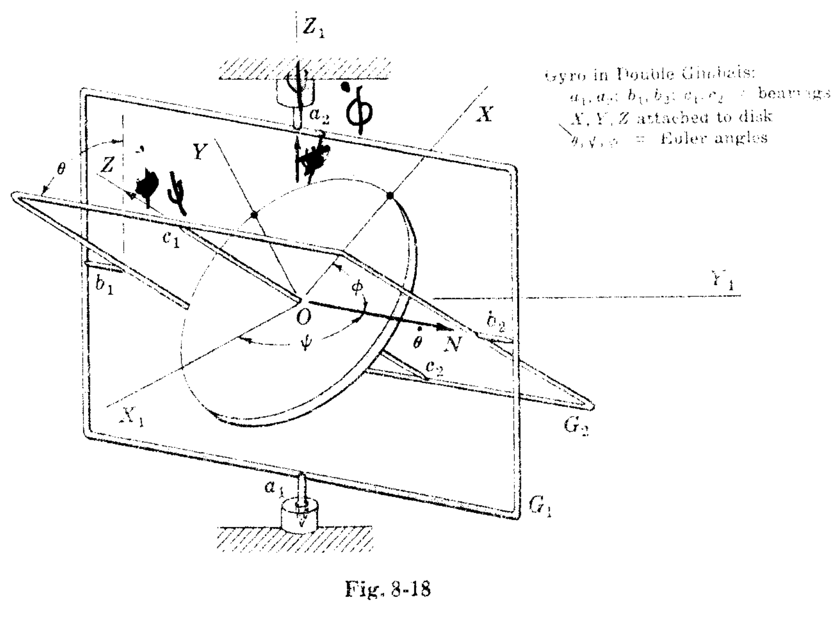


Fig. 8-18

Pg 365

4/25 Lecture

Euler and Hamilton

- Sloan wants to buy book $\Rightarrow$ Drop by wifes office, she will put you in touch.
- Drop test – I will drop one test and one homework but not the last test or HWK.
- Presentation

Section - Kinetic Energy 3-D Bodies

Recall, in terms of the principle moments

$$T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$$

If the $Oxyz$ and $O123$ coordinate systems coincide when $\phi, \theta, \psi = 0$. Then the angular velocities can be written in terms of the Euler angles

$$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$

$$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$$

$$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi}$$

Example - Consider system with two Euler angles constrained

$$\dot{\psi} = \text{constant} \equiv \Omega$$

$$\dot{\phi} = \text{constant} \equiv \gamma$$

want EOM for θ.

$$V = 0$$

$$T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$$

$$= \frac{1}{2} I_1 \left[\gamma \sin\theta \sin\Omega t + \dot{\theta} \cos\Omega t \right]^2$$

$$+ \frac{1}{2} I_2 \left[\gamma \sin\theta \cos\Omega t - \dot{\theta} \sin\Omega t \right]^2$$

$$+ \frac{1}{2} I_3 \left[\gamma \cos\theta + \Omega \right]^2 = L$$

$$\frac{\partial L}{\partial \theta} = I_1 \left[\gamma \sin\theta \sin\Omega t + \dot{\theta} \cos\Omega t \right] \gamma \sin\Omega t \cos\theta$$

$$+ I_2 \left[\qquad \right] (+\gamma \cos\theta \cos\Omega t)$$

$$+ I_3 \left[\quad \right] \gamma (-\sin\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = I_1 [\;] \cos \Omega t + I_2 [\;] (-\sin \Omega t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} =$$

You get idea, pretty messy but you get eom in terms of fixed space angles mechanically.

Section Hamilton

Hamiltonian $H = \sum \dot{q}_i p_i - L$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Hamiltonian is Total energy

$$\boxed{H = T + V}$$

$H(q_i, p_i)$ where $L(q_i, \dot{q}_i)$

$\Rightarrow$ In one dimension $2T = \cancel{4}\, \dot{x} p = m\dot{x}^2$

$$H = 2T - (T - V) = T + V$$

$\Rightarrow$ Same kind of transformation we carry out in thermodynamic to go from $E \to F$

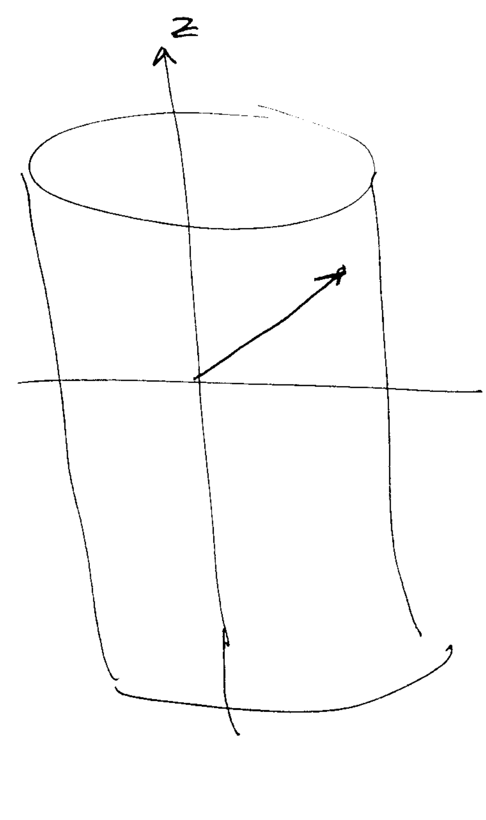
Hamilton's Equations of Motion

$$\frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i$$

Two first order diff eqn rather than 1 second order

Example – Particle confined to surface of cylinder, subject to Hooke's Law force $\vec{F} = -k\vec{r}$ directed toward origin.



$$V = \frac{1}{2}k\left(r^2 + z^2\right)$$

$$= \frac{1}{2}kr^2 + \frac{1}{2}kz^2$$

constant

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right)$$

0

$$L = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m \dot{z}^2 - \frac{1}{2} k r^2 - \frac{1}{2} k z^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m \dot{z}$$

Generalized Momenta

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$H = T + V = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m \dot{z}^2 + \frac{1}{2} k z^2 + \frac{1}{2} k r^2$$

$\Longrightarrow$ Must rewrite in terms of generalized momenta.
$H(p_z, p_\theta, z)$

$$H = \frac{1}{2} \frac{p_\theta^2}{m r^2} + \frac{1}{2} \frac{p_z^2}{m} + \frac{1}{2} k z^2 + \frac{1}{2} k r^2$$

EOM

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m r^2} = \dot{\theta}$$

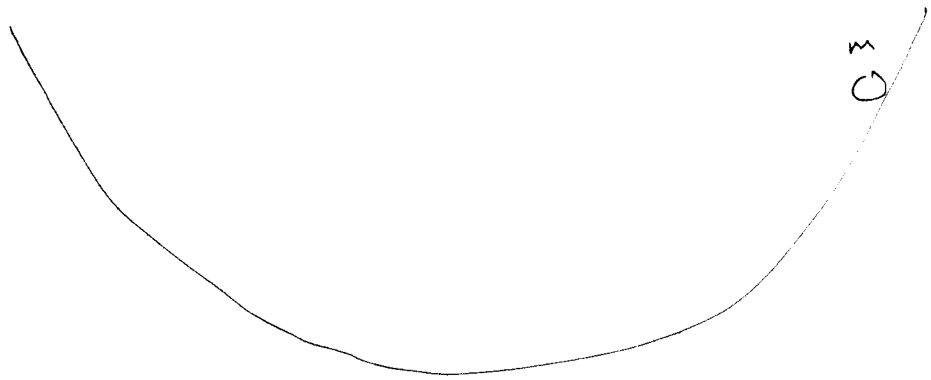
$$\frac{\partial H}{\partial p_z} = \frac{p_z}{m} = \dot{z}$$

$$\frac{\partial H}{\partial \theta} = 0 = -\dot{p}_\theta$$

$$\frac{\partial H}{\partial z} = k z = -\dot{p}_z$$

Newton III

Section - <u>Justin's Bowl</u>



$$V(x) = mgy = mg\alpha x^2$$

$$v = \dot{s} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\dot{x} = \left(\sqrt{1+4\alpha^2 x^2}\right)\dot{x}$$

<u>Arc Length</u> (s) $\qquad ds^2 = dx^2 + dy^2$

$$ds = \sqrt{dx^2+dy^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(1+4\alpha^2x^2\right)\dot{x}^2$$

$$L = T - V = \frac{1}{2}m\left(1+4\alpha^2x^2\right)\dot{x}^2 - mg\alpha x^2$$

$$\frac{\partial L}{\partial x} = \frac{1}{2}m\dot{x}^2\left(8\alpha^2 x\right) - 2mg\alpha x$$

$$\frac{\partial L}{\partial \dot{x}} = m\left(1+4a^2x^2\right)\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 8ma^2 x\dot{x}^2 + m\left(1+4a^2x^2\right)\ddot{x}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$\frac{1}{2}m\dot{x}^2\left(8a^2x\right) - 2mg\alpha x - 8ma^2x\dot{x}^2 - m\left(1+4a^2x^2\right)\ddot{x} = 0$$

<u>Well that's a mess</u> Find small amplitude behavior,

$\Rightarrow$ Look for linear terms

$$-2mg\alpha x - m\ddot{x} = 0$$

$$\ddot{x} + 2g\alpha x = 0 \qquad x(t) = A\cos(\omega t + \phi)$$

$$\omega = \sqrt{2g\alpha}$$