Lecture 2/17/2003

Constrained Motion

# Section 1 - Expansion

When and how to use a power series expansion as an approximation

**When**

1. When the exact system cannot be solved or is very difficult to solve.
2. When the exact ~~system~~ solution is too complicated to yield insight.
3. When the problem uses the word approximate or small.

**How** — Find a small parameter $x \ll 1$, then look up series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots$$

$$x^3 \ll x^2 \ll x$$

and higher order terms can be ignored.

> A power series is the mathematically correct way to form an approximation to a system.

**Example**

$$\frac{1}{r_e + z}$$

$r_e$ = radius of earth

$z$ = height above surface

$$\frac{1}{r_e+z} = \frac{1}{r_e}\left(\frac{1}{1+z/r_e}\right) = \frac{1}{z}\left(\frac{1}{\frac{r_e}{z}+1}\right)$$

Taylor Expansion (Look it up)

$$\frac{1}{1+x} = 1 - x + x^2 + \ldots$$

## Two Choices

$$\frac{1}{r_e}\,\frac{1}{1 + z/r_e} = \frac{1}{r_e}\left(1 - \frac{z}{r_e} + \left(\frac{z}{r_e}\right)^2 + \cdots\right) \qquad (A)$$

$$\frac{1}{z}\left(\frac{1}{r_e/z + 1}\right) = \frac{1}{z}\left(1 - \frac{r_e}{z} + \left(\frac{r_e}{z}\right)^2 + \cdots\right) \qquad (B)$$

$$(A) \neq (B)$$

Oh-oh — (A) is a good approximation when

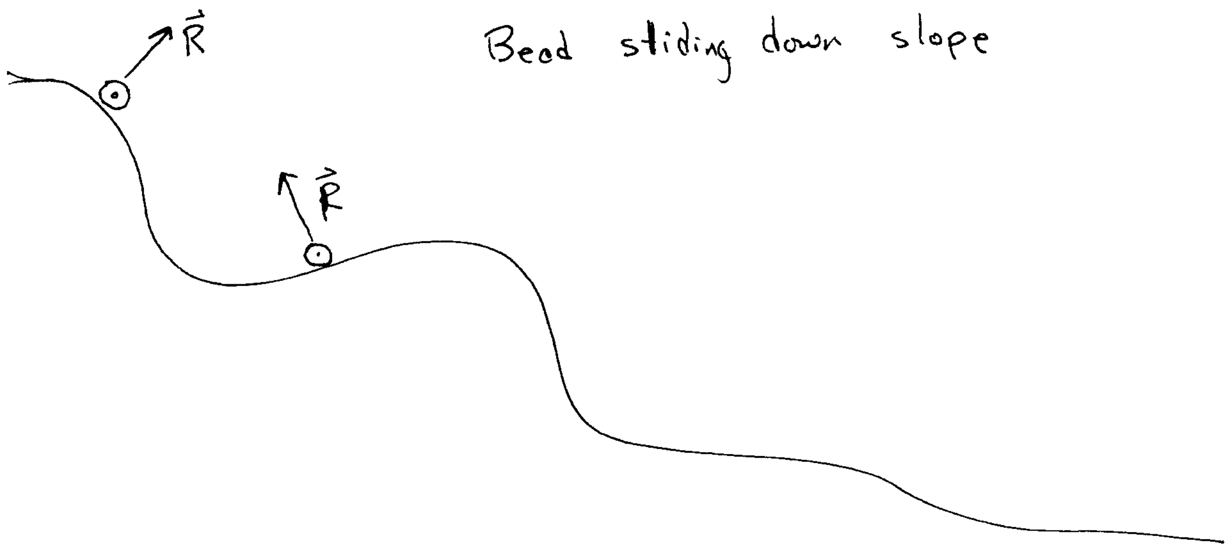
$z/r_e \ll 1 \Rightarrow$ near the earth.

(B) is a good approximation when

$r_e/z \ll 1$ far from the earth.

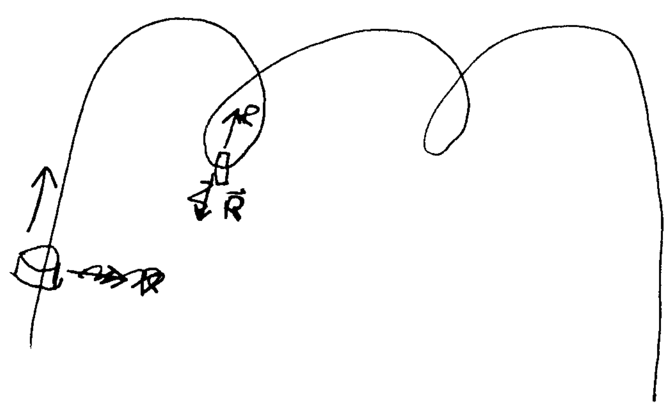
# Section 2 - Constrained Motion

Bead sliding down slope




<u>Forces of Constraint</u> :- A force that keeps a particle confined to a surface or trajectory.

## Bead Confined to Wire




Force of constraint provide the necessary force to keep the bead from accelerating through the surface or off the wire $\Longrightarrow$ Force depends on motion.

~~Make~~ Work in a coordinate system along the surface of ~~bead~~, $\vec{v}$ is $\parallel$ to surface.

Forces of Constraint ($\vec{R}$) are normal (perpendicular) to surface

$\Rightarrow$ Forces of Constraint do not do work

$$\text{Power} = \vec{R} \cdot \vec{v} = 0$$

$\Rightarrow$ If surfaces (wires) are frictionless, the forces of constraint do not enter the energy equations.

$$E_{sys} = \frac{1}{2} m \dot{s}^2 + V(x, y, z) = \text{constant}$$

## Section 3 – Arc Length and Curvature

In two dimension, $y(x) = \alpha x^2$

Arc Length (s) $\quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

### Curvature ($K$)

$$K = \frac{\frac{d^2y}{dx^2}}{\left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right)^3} = \frac{\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}$$

### Radius of Curvature ($\frac{1}{K}$)

**Example** For $y = \alpha x^2$,

$$\frac{dy}{dx} = 2\alpha x \qquad \frac{dy^2}{dx^2} = 2\alpha$$

$$K = \frac{2\alpha}{\left(\sqrt{1 + 4\alpha^2 x^2}\right)^3}$$

$$\frac{1}{K} = \frac{\left(\sqrt{1 + 4\alpha x^2}\right)^3}{2\alpha} = R$$

To evaluate the force of constraint, solve

Newton II ~~tangent~~ normal to curve.

<u>Centripetal Acceleration</u> $\quad -\frac{v^2}{r} = -\frac{\dot{s}^2}{r}$

Points toward center.

<u>Tangent to Curve</u>

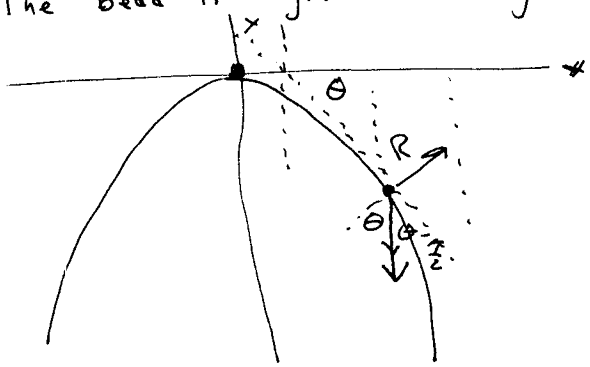
$$\frac{mv^2}{r} = R + F_n$$

# Section 4 - Example

Let mass $m$ slide down parabolic surface, $y = -\alpha x^2$

at what value of $x$ does the bead leave to surface.

The bead is given a slight ~~k~~nudge at the top of the curve




$$V(x) = mgy = -\alpha m g x^2$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{s}^2$$

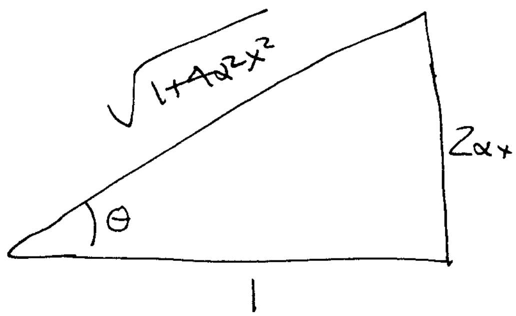
$$E_{sys} = 0 = \frac{1}{2} m v^2 - \alpha m g x^2 = T_0 + V(0)$$

$$v = \sqrt{2\alpha g x^2} = \sqrt{2\alpha g}\ x$$

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \tan\theta = \frac{dy}{dx} = 2\alpha x$$

$$\boxed{-\frac{mv^2}{r} = R - mg\cos\theta}$$

$$\boxed{\text{Bead will leave surface when } R=0.}$$




$$\tan\theta = 2\alpha x$$

$$\cos\theta = \frac{1}{\sqrt{1+4\alpha^2x^2}}$$

$$-\frac{mv^2}{r} = R - \frac{mg}{\sqrt{1+4\alpha^2x^2}}$$

$r \equiv$ <u>Radius of Curvature</u>

$$K = \frac{\frac{d^2y}{dx^2}}{\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right)^3} \qquad \frac{dy}{dx} = -2\alpha x \qquad \frac{d^2y}{dx^2} = -2\alpha$$

$$= \frac{-2\alpha}{\left(\sqrt{1+4\alpha^2x^2}\right)^3}$$

$$r = \left|\frac{1}{K}\right| = \frac{\left(\sqrt{1+4\alpha^2x^2}\right)^3}{2\alpha}$$

$$R = \frac{mg}{\sqrt{1+4a^2x^2}} - \frac{mv^2}{r}$$

$$= \frac{mg}{\sqrt{1+4a^2x^2}} - \frac{m(2\alpha g x^2)}{\left[\frac{\left(\sqrt{1+4a^2x^2}\right)(1+4a^2x^2)}{2\alpha}\right]} = 0$$

Cancel $\dfrac{mg}{\sqrt{1+4a^2x^2}}$

$$1 - \frac{(2\alpha)^2 x^2}{1+4a^2x^2} = 0$$

$$1+4a^2x^2 - 4a^2x^2 = 0$$

No solution $\Longrightarrow$ Bead Never leaves parabola