Homework 2

Due Monday 1/21/2010 - at beginning of class

Reading Assignment - Chapter 2

Griffiths' Problems

1.43

2.6

2.15

2.17

2.20

2.23

2.30 Parts (b) and (c)

2.31

2.37

2.38

2.39

2.44

2.46 Complete Part(a) only. State the equation you would use to do Part (b). Skip Part (c).

1.43

(a)

$x = 3$ is in the integral range

$$\int_2^6 (3x^2 - 2x - 1)\,\delta(x-3)\,dx = \underbrace{3(3)^2}_{27} \underset{-6}{-2\cdot 3} \underset{-1}{-1}$$

$$= 20$$

(b) $\pi \in [0, 5]$ so

$$\int_0^5 \cos x\,\delta(x-\pi)\,dx = \cos\pi = -1$$

(c) $-1 \notin [0, 3]$

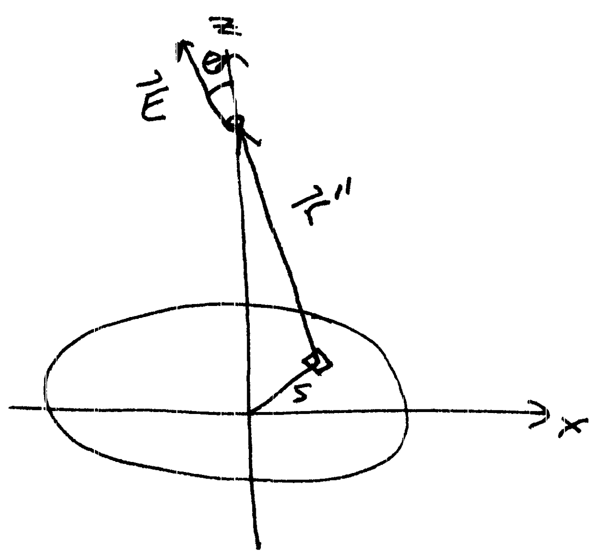
$$\int = 0$$

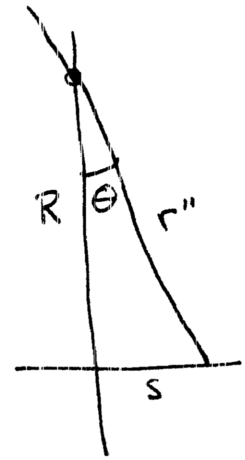
(d) $x = -2 \in [-\infty, \infty]$

$$\int_{-\infty}^{\infty} \ln(x+3)\,\delta(x+2)\,dx = \ln(-2+3) = \ln(1)$$

$$= 0$$

2.6





By symmetry, the x, y components cancel leaving only the z component.

The z-component of the small piece of charge dq drawn is

$$|d\vec{E}| = \frac{k\,dq}{r''^2}\cos\theta$$

$$\cos\theta = \frac{R}{r''} \qquad dq = \sigma\, ds\, s\, d\phi$$

$$|d\vec{E}| = \frac{k\sigma s\, ds\, d\phi\, R}{r''^3}$$

$$r'' = \sqrt{s^2 + R^2}$$

$$|\vec{E}| = \int_0^a ds \int_0^{2\pi} d\theta \; \frac{k\sigma R \, s}{(s^2+R^2)^{3/2}}$$

$$= 2\pi k\sigma R \int_0^a \frac{ds\, s}{(s^2+R^2)^{3/2}}$$

$$= 2\pi k\sigma R \left(\frac{1}{R} - \frac{1}{\sqrt{a^2+R^2}} \right)$$

If we let $R \to z$, some point on $+z$ axis

$$\vec{E} = 2\pi k\sigma z \left(\frac{1}{z} - \frac{1}{\sqrt{a^2+z^2}} \right) \hat{z}$$

$$= \frac{\sigma}{2\varepsilon_0} z\hat{z} \left(\frac{1}{z} - \frac{1}{\sqrt{a^2+z^2}} \right)$$

If $a \to \infty$, $\vec{E} \to \frac{\sigma}{2\varepsilon_0}\hat{z}$ the field of an infinite plane. ✓

If $z \to \infty$, $(a^2+z^2)^{-1/2} = \frac{1}{z}\left(1 + \frac{a^2}{z^2}\right)^{-1/2}$

$$= \frac{1}{z}\left(1 - \frac{1}{2}\frac{a^2}{z^2} + \cdots\right) \quad \text{binomial thm}$$

$$\frac{1}{z} - \left(a^2+z^2\right)^{-1/2} \longrightarrow \frac{1}{z} - \frac{1}{z} + \frac{1}{2}\frac{a^2}{z^3} + \cdots$$

$$= \frac{1}{2}\frac{a^2}{z^3}$$

$$\vec{E} \rightarrow \frac{\sigma z}{2\varepsilon_0} \cdot \frac{1}{2}\frac{a^2}{z^3}\,\hat{z}$$

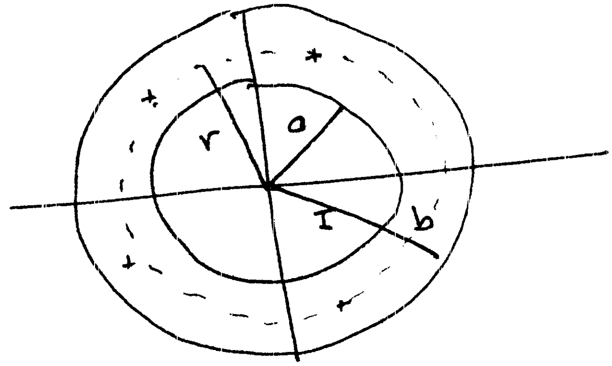
$$= \frac{a^2\sigma}{4\varepsilon_0 z^2}\,\hat{z} = \cancel{\pi a^2 \sigma}$$

$$= \frac{\pi a^2 \sigma}{4\pi\varepsilon_0 z^2}\,\hat{z} = \frac{Q}{4\pi\varepsilon_0 z^2}\,\hat{z}$$

✓ Field of point charge

2.15

$$\rho = \frac{k}{r^2}$$



The charge enclosed in a Gaussian surface with $a < r < b$ drawn above

$$Q_{enc} = \int \rho \, d\tau$$

$$= \int_a^r 4\pi r^2 \rho \, dr$$

$$= \int_a^r 4\pi r^2 \cdot \frac{k}{r^2} \, dr \quad] \text{ shell method}$$

$$= 4\pi k (r - a)$$

The total charge of the shell is

$$Q_T = 4\pi k (b - a)$$

Inside all charge $\quad r < a$

$$\vec{E} = 0$$

Outside all charge $\quad r > b$

$$\vec{E} = \frac{Q_T}{4\pi\varepsilon_0 r^2}\hat{r}$$

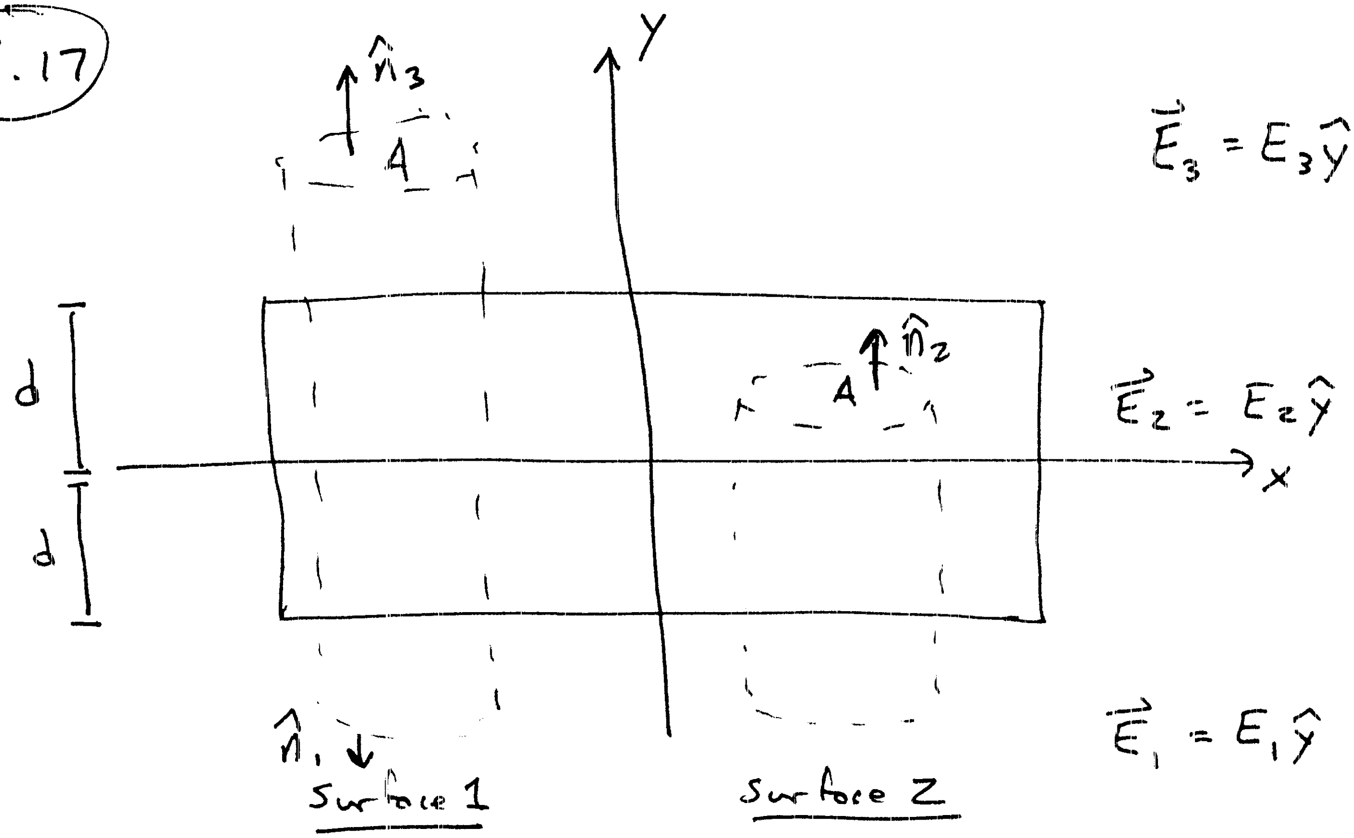
field of point charge with total charge of system.

In volume charge $\quad a < r < b$

$$\Phi = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} \quad \text{Gauss Law}$$

$$\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{4\pi k(r-a)}{4\pi\varepsilon_0 r^2}\hat{r}$$

2.17



Surface 1 $\quad Q_{enc} = 2dA\rho \qquad \hat{n}_3 = \hat{y} \qquad \hat{n}_1 = -\hat{y}$

$$\Phi = \vec{E}_3 \cdot \hat{n}_3 A + \vec{E}_1 \cdot \hat{n}_1 A$$

$$= E_3 A + (-E_1 A) = \frac{Q_{enc}}{\varepsilon_0} \qquad \text{Gauss}$$

By symmetry $E_3 = -E_1$

$$2E_3 A = \frac{Q_{enc}}{\varepsilon_0} = \frac{2dA\rho}{\varepsilon_0}$$

$$E_3 = \frac{d\rho}{\varepsilon_0} = \frac{d\rho}{\varepsilon_0}$$

$$\vec{E}_3 = \frac{\rho d}{\varepsilon_0}\hat{y} \qquad \vec{E}_1 = -\frac{\rho d}{\varepsilon_0}\hat{y}$$

Surface 2 $\qquad Q_{enc} = \rho A(y+d)$

$$\Phi = \vec{E}_2 \cdot \hat{n}_2 A + \vec{E}_1 \cdot \hat{n}_1 A$$

$$= E_2 A - E_1 A = \frac{Q_{enc}}{\varepsilon_0} = \frac{\rho A(y+d)}{\varepsilon_0}$$

$$E_2 = E_1 + \frac{\rho(y+d)}{\varepsilon_0}$$

$$= -\frac{\rho d}{\varepsilon_0} + \frac{\rho(y+d)}{\varepsilon_0} = \frac{\rho y}{\varepsilon_0}$$

$$\vec{E}_2 = \frac{\rho y}{\varepsilon_0}\hat{y}$$

(2.20) To be possible, the field must be curl free.

(a)

$$\nabla \times \vec{E} = k \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix}$$

$$= \hat{x}(0 - 2y) \neq 0$$
$$+ \hat{y}(\ )$$
$$+ \hat{z}(\ )$$

not an electric field

(b)

$$\nabla \times \vec{E} = k \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & (2xy + z^2) & 2yz \end{vmatrix}$$

$$= \left[\hat{x}(2z - 2z) - \hat{y}(0 - 0) + \hat{z}(2y - 2y)\right] k$$

$= 0$ can be electric field.

$$\vec{E} = -\nabla V$$

$$\vec{E} \cdot d\vec{l} = -\frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy - \frac{\partial V}{\partial z} dz$$

$$\frac{\partial V}{\partial x} = -ky^2$$

$$V = -kxy^2 + f(y,z)$$ where f is some function.

$$\frac{\partial V}{\partial y} = -k(2xy + z^2) = -2kxy + \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = -kz^2$$

$$f = -kyz^2 + g(z)$$

$$V = -kxy^2 - kyz^2 + g(z)$$

$$\frac{\partial V}{\partial z} = -2kyz = -2kyz + \frac{\partial g}{\partial z}$$

$g =$ constant

$$V = -kxy^2 - kyz^2 + C$$

$$\vec{E} = -\nabla V = \left(+ky^2,\ 2kxy + kz^2,\ 2kyz\right) \checkmark$$

2.28

$$\vec{E} = \begin{cases} 0 & r<a \\ \dfrac{k(r-a)}{r^2\varepsilon_0}\,\hat{r} & a<r<b \\ \dfrac{Q_T}{4\pi\varepsilon_0 r^2}\,\hat{r} & r>b \end{cases}$$

$$V = -\int E\,dr$$

$$V = \frac{Q_T}{4\pi\varepsilon_0 r} + C_{III} \quad (r > b)$$, point charge field

$$V = C_I \qquad r<a$$, since $E=0$

$$V = -\frac{k}{\varepsilon_0}\int\left(\frac{1}{r} - \frac{a}{r^2}\right)dr \qquad a<r<b$$

$$= -\frac{k}{\varepsilon_0}\left(\ln(r) + \frac{a}{r}\right) + C_{II}$$

where C_I, C_{III}, C_{II} are constants of integration.

The potential must be continuous.

Select C_{III} so $V(\infty)=0 \Rightarrow C_{III}=0$

$$V(b) = -\frac{k}{\varepsilon_0}\left(\ln(b) + \frac{a}{b}\right) + C_{II} = \frac{Q_T}{4\pi\varepsilon_0 b}$$

$$C_{II} = \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln(b) + \frac{k}{\varepsilon_0}\frac{a}{b}$$

$$V(r) = \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\left(\ln(b) - \ln(r)\right) + \frac{ka}{\varepsilon_0}\left(\frac{1}{b} - \frac{1}{r}\right)$$

$$= \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln\left(\frac{b}{r}\right) + \frac{ka}{\varepsilon_0 br}(r-b)$$

$a < r < b$

Match potential at $r = a$

$$V(a) = C_I = \frac{Q_T}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right) + \frac{ka}{\varepsilon_0 ba}(a-b)$$

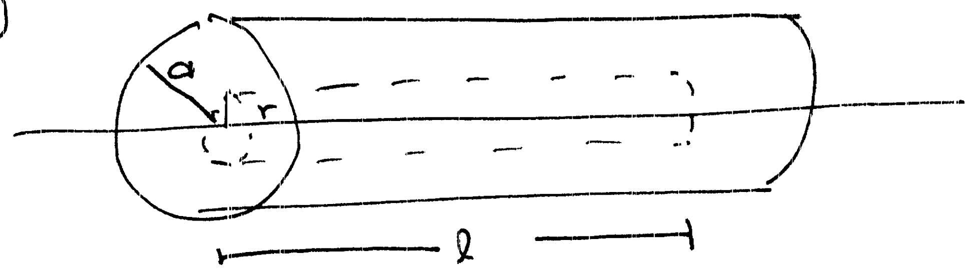
$$Q_T = 4\pi k (b-a)$$

$$V(a) = \frac{4\pi k (b-a)}{4\pi\varepsilon_0 b} + \frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right) + \frac{k}{\varepsilon_0 b}(a-b)$$

$$= \frac{k}{\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

2.30

(b)



Cylindrical Gaussian Surface radius r and length l.

$$\Phi = 2\pi r l E = \frac{Q_{enc}}{\varepsilon_0}$$

For $r<a$, $Q_{enc} = 0$, $E = 0$.

For $r>a$, $Q_{enc} = 2\pi a l \sigma$

$$E = \frac{Q_{enc}}{2\pi r l \varepsilon_0} = \frac{2\pi a l \sigma}{2\pi r l \varepsilon_0}$$

$$= \frac{a\sigma}{r\varepsilon_0}$$

B.C. $E_2 - E_1 = \frac{\sigma}{\varepsilon_0} = \frac{a\sigma}{a\varepsilon_0} - 0 = \frac{\sigma}{\varepsilon_0}$ ✓

(c) From example 2.7

$$V_{out}(z) = \frac{R^2\sigma}{\varepsilon_0 z} \qquad \text{outside}$$

$$V_{in}(z) = \frac{R\sigma}{\varepsilon_0} \qquad \text{inside}$$

2.34 Continuous

$$V_{out}(R) = \frac{R^2\sigma}{\varepsilon_0 R} = \frac{R\sigma}{\varepsilon_0} = V_{in}(R) = \frac{R\sigma}{\varepsilon_0}$$

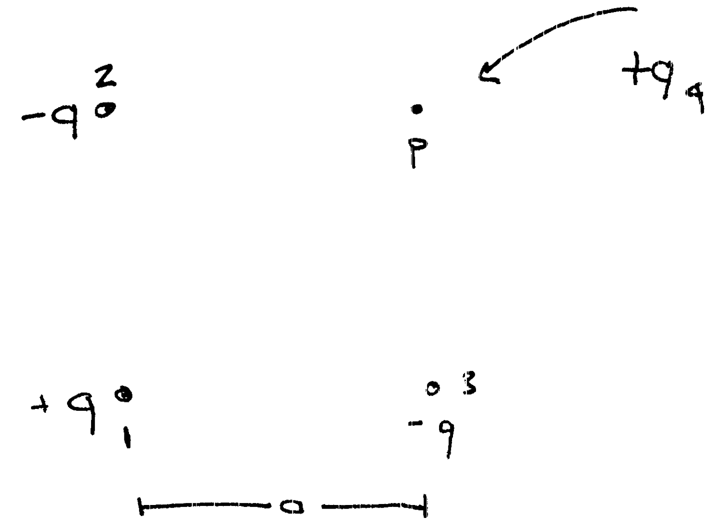
2.36

$$\left.\frac{\partial V_{out}}{\partial z}\right|_R - \left.\frac{\partial V}{\partial z}\right|_{in} = -\frac{\sigma}{\varepsilon_0}$$

$$-\left.\frac{R^2\sigma}{\varepsilon_0 z^2}\right|_R - 0 = -\frac{\sigma}{\varepsilon_0} = -\frac{\sigma}{\varepsilon_0} \checkmark$$

(2.31)

(a)



$$W = q V_p$$

$$= q\left(\frac{-kq}{a} + \frac{-kq}{a} + \frac{kq}{\sqrt{2}a} \right)$$

$$= \frac{kq^2}{a}\left(\frac{1}{\sqrt{2}} - 2 \right)$$

(b) The work to build the system is

$$W_1 = 0$$

$$W_2 = -\frac{kq^2}{a}$$

$$W_3 = -q\left(\frac{kq}{a} + \frac{-kq}{\sqrt{2}a} \right) = \frac{kq^2}{a}\left(\frac{1}{\sqrt{2}} - 1 \right)$$

$$W_4 = q\left(\frac{-kq}{a} + \frac{-kq}{a} + \frac{kq}{\sqrt{2}\,a} \right)$$

$$= \frac{kq^2}{a}\left(\frac{1}{\sqrt{2}} - 2 \right)$$

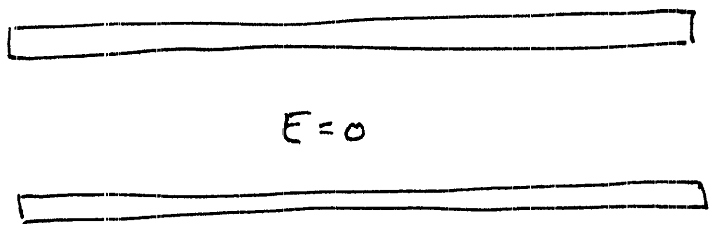
$$W = W_1 + W_2 + W_3 + W_4$$

$$= 0 + \frac{kq^2}{a}\left(-1 + \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}} - 2 \right)$$

$$= \frac{kq^2}{a}\left(\frac{2}{\sqrt{2}} - 4 \right) = \frac{2kq^2}{a}\left(\frac{1}{\sqrt{2}} - 2 \right)$$

2.37

$$E = \frac{\sigma}{\varepsilon_0}$$



$\sigma = \frac{Q}{A}$

$E = 0$

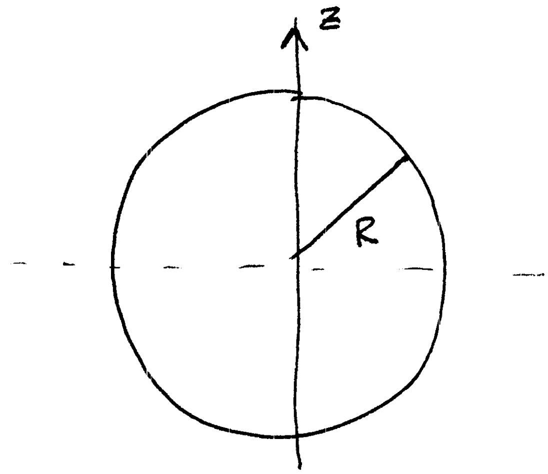
$\sigma = Q/A$

$$E = \frac{\sigma}{\varepsilon_0}$$

$$P = \sigma \frac{E}{2} = \frac{\sigma \cdot \sigma/\varepsilon_0}{2}$$

$$= \frac{\sigma^2}{2\varepsilon_0} = \frac{Q^2}{2A^2\varepsilon_0}$$

2.38



Electric field $\vec{E} = 0 \qquad r < R$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \qquad r > R$$

Electric Pressure

$$\vec{P} = \sigma \vec{E}_{ave} = \frac{\sigma}{2}\vec{E} \qquad r > R$$

$$= \frac{\sigma Q}{8\pi\epsilon_0 R^2}\hat{r} \qquad \sigma = \frac{Q}{4\pi R^2}$$

Total force exerted on northern hemisphere

$$\vec{F} = \int \vec{P}\, da$$

$$da = R\,d\theta \sin\theta\, R\,d\phi = R^2 \sin\theta\, d\theta\, d\phi$$

Evidently only $\hat{z}$ component survives,

$$\hat{r} = \underbrace{\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y}}_{\text{integrates to zero}} + \cos\theta\,\hat{z}$$

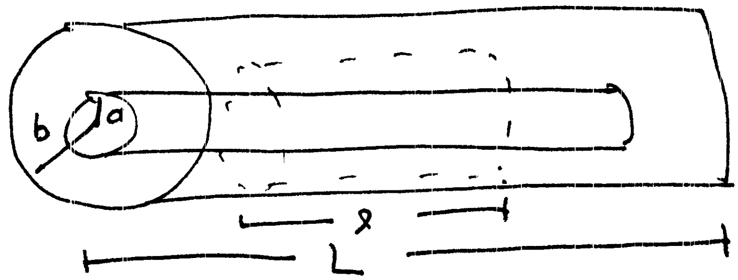
$$\vec{F} = \int \vec{P}\,da = \int \left(\frac{\sigma Q}{4\pi\varepsilon_0 R^2} \cdot \cos\theta\,\hat{z}\right)\left(R^2 \sin\theta\, d\theta\, d\phi\right)$$

$$= \frac{\sigma Q}{8\pi\varepsilon_0} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi/2} d\theta \cos\theta \sin\theta}_{\frac{1}{2}}\ \hat{z}$$

$$= \frac{\sigma Q\,\pi}{8\pi\varepsilon_0}\hat{z} = \frac{\sigma Q}{8\varepsilon_0}\,\hat{z}$$

$$= \frac{Q}{8\varepsilon_0}\left(\frac{Q}{4\pi R^2}\right) = \frac{Q^2}{32\pi\varepsilon_0 R^2}\,\hat{z}$$

2.39



Add a charge Q to inner conductor, and charge −Q to outer conductor.

Use a cylindrical Gaussian surface that encloses the inner conductor.

$$Q_{enc} = \lambda l$$

where $\lambda = Q/L$

Gauss Law

$$\Phi = 2\pi r l E = \frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi r \varepsilon_0}$$

Potential Difference

$$V = -\int_a^b E\,dr$$

$$= \frac{-\lambda}{2\pi\varepsilon_0} \ln(b/a) = -\frac{Q}{2\pi\varepsilon_0 L} \ln(b/a)$$

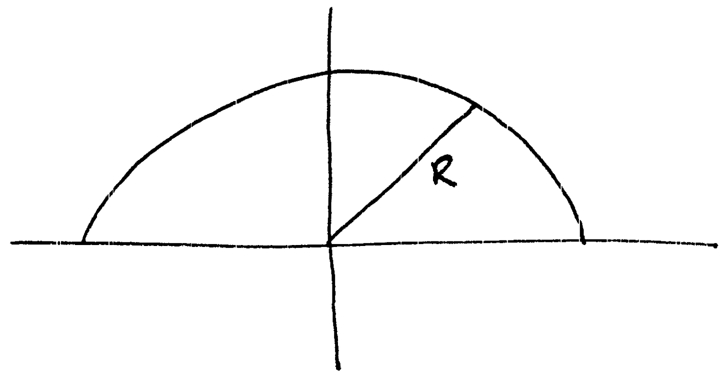
Capacitance

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{\frac{Q}{2\pi\varepsilon_0 L} \ln(b/a)}$$

$$= \left(\frac{2\pi\varepsilon_0}{\ln(b/a)}\right) L$$

Capacitance per unit length

2.44



Center

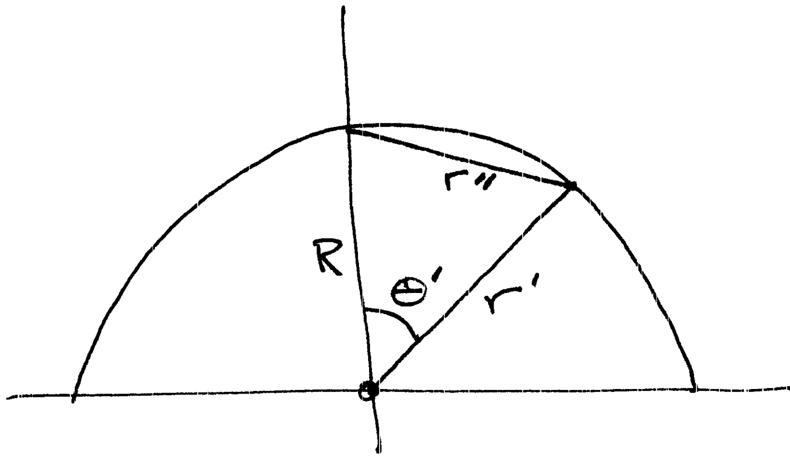
$$V(0) = \int \frac{k\sigma\,da}{R}$$

$$= \frac{k\sigma}{R} \cdot \text{surface area}$$

$$= \frac{k\sigma}{R} \cdot 2\pi R^2 = 2\pi k\sigma R$$

$$= \frac{\sigma R}{2\varepsilon_0}$$

Pole



$$V_{pole} = \int \frac{k\sigma da'}{r'} \qquad r' = R\hat{r}'$$

Low of Cosines

$$r''^2 = R^2 + R^2 - 2R^2\cos\theta$$

$$= 2R^2(1-\cos\theta')$$

$$V_{pole} = \frac{k\sigma}{\sqrt{2}R}\int_0^{2\pi} d\phi' \int_0^{\pi/2} d\theta' \frac{R^2\sin\theta'}{\sqrt{1-\cos\theta'}}$$

$$= \frac{2\pi k\sigma R}{\sqrt{2}} \underbrace{\int_0^{\pi/2} d\theta' \frac{\sin\theta'}{\sqrt{1-\cos\theta'}}}_{2}$$

Maple

$$V_{pole} = \frac{4\pi k \sigma R}{\sqrt{2}} = \frac{\sigma R}{\sqrt{2}\,\varepsilon_0}$$

$$V_{pole} - V_{center} = \frac{\sigma R}{\sqrt{2}\,\varepsilon_0} - \frac{\sigma R}{2\varepsilon_0}$$

2.46

(a) $\vec{E} = -\nabla V = -\nabla\left(A\frac{e^{-\lambda r}}{r}\right)$

$$= \frac{A\lambda e^{-\lambda r}}{r}\hat{r} - Ae^{-\lambda r}\nabla\left(\frac{1}{r}\right)$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$$ pg 50

$$\vec{E} = \frac{A\lambda}{r}e^{-\lambda r}\hat{r} + \frac{Ae^{-\lambda r}}{r^2}\hat{r}$$

$$= \frac{Ae^{-\lambda r}}{r^2}(1+\lambda r)\hat{r}$$

(b) Charge density

$$\nabla\cdot\vec{E} = \rho/\epsilon_0$$