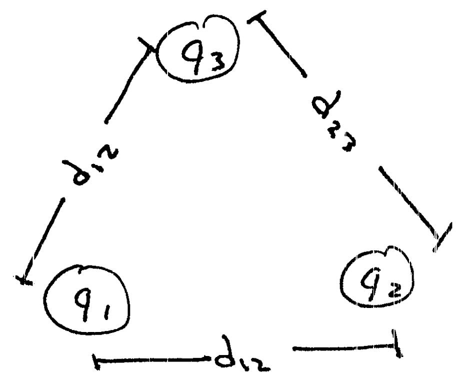
Electrostatic Energy

Consider our three point charges again,



We can ask two different things about the energy of the system.

I. <u>Total Energy of System</u> — This is the energy that would be recovered as KE if the charges were allowed to move infinitely far apart. Conversely, it is the total <u>work</u> required to build the system charge by charge.

II. <u>Potential Energy of q_3</u> — This is the kinetic energy that would be recovered if q_3 were replaced while q_1 and q_2 are held fixed.

The potential energy of q_3 with q_1 and q_2 fixed

is

$$U_3 = q_3 V(\vec{r}_3)$$

$$= q_3 \left(\frac{k q_1}{d_{13}} + \frac{k q_2}{d_{23}} \right)$$

If q_3 was released,

$$\Delta U = \Delta KE$$

the final kinetic energy would be

$$U_3 = KE_f = \frac{1}{2} m v^2$$

Conversely, the work to move q_3 into position is

$$W = U_3$$

The total energy of the system of charge is the work to build it piece by piece.

The work to place the first charge with no other charges present is

$$W_1 = 0$$

The work to place q_2 in the presence of q_1 is

$$W_2 = q_2 V_1(\vec{r}_2)$$

$$= q_2 \left(\frac{k q_1}{d_{12}} \right)$$

The work to place q_3 with q_1 and q_2 in place is calculated above

$$W_3 = q_3 \left(\frac{k q_1}{d_{13}} + \frac{k q_2}{d_{23}} \right)$$

The total energy of the system is

$$U = W_1 + W_2 + W_3$$

$$= \frac{k\, q_1 q_2}{d_{12}} + \frac{k\, q_1 q_3}{d_{13}} + \frac{k\, q_2 q_3}{d_{23}}$$

$$= \frac{1}{2} \sum_i \sum_{i \neq j} \frac{k\, q_i q_j}{d_{ij}}$$

$$= \frac{1}{2} \sum_i q_i\, V(\vec{r}_i) \qquad = W$$

where $V(\vec{r}_i)$ is the potential at $\vec{r}_i$ due to all charges except q_i

$$V(\vec{r}_i) = \sum_{\substack{i \neq j \\ j = 1 \ldots N}} \frac{k\, q_j}{d_{ij}}$$

We can extend this to continuous distributions by working with infinitesimal charges $dq = \rho\, d\tau$

$$W = U = \frac{1}{2}\int V\, dq$$

$$= \frac{1}{2}\int \rho V\, d\tau$$

I would prefer the energy in terms of the fields not the charges.

Use Gauss $\nabla \cdot \vec{E} = \rho/\varepsilon_0 \Rightarrow \rho = \varepsilon_0 \nabla \cdot \vec{E}$

$$W = \frac{1}{2}\varepsilon_0 \int (\nabla \cdot \vec{E})\, V\, d\tau$$

<u>Front cover</u> $\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla f$

$$f \rightarrow V \quad \vec{A} \rightarrow \vec{E}$$

$$\nabla \cdot (V\vec{E}) = V(\nabla \cdot \vec{E}) + \vec{E} \cdot \nabla V$$

$$V(\nabla\cdot\vec{E}) = \nabla\cdot(V\vec{E}) - \vec{E}\cdot\nabla V$$

$$W = \frac{1}{2}\int \rho V \, d\tau = \frac{1}{2}\varepsilon_0 \int V(\nabla\cdot\vec{E})\, d\tau$$

$$= \frac{1}{2}\varepsilon_0 \int \left(\nabla\cdot(V\vec{E}) - \vec{E}\cdot\nabla V\right) d\tau$$

$$-\nabla V = \vec{E}$$

$$W = \frac{1}{2}\varepsilon_0 \underbrace{\int \nabla\cdot(V\vec{E})\, d\tau}_{} + \frac{1}{2}\varepsilon_0 \int \vec{E}\cdot\vec{E}\, d\tau$$

divergence thm

$$\int_V \nabla\cdot(V\vec{E})\, d\tau = \int_S V\vec{E}\cdot d\vec{a}$$

Let $S \to \infty$, if charge compact $\vec{E} \to 0$ as $\frac{1}{r^2}$

and $V \to 0$ as $\frac{1}{r}$ so $\int_S \to 0$

$$W = \frac{1}{2}\varepsilon_0 \int_{\text{space}} \vec{E}\cdot\vec{E}\, d\tau$$

Energy density of electric field (u)

$$u = \frac{1}{2}\varepsilon_0 E^2$$