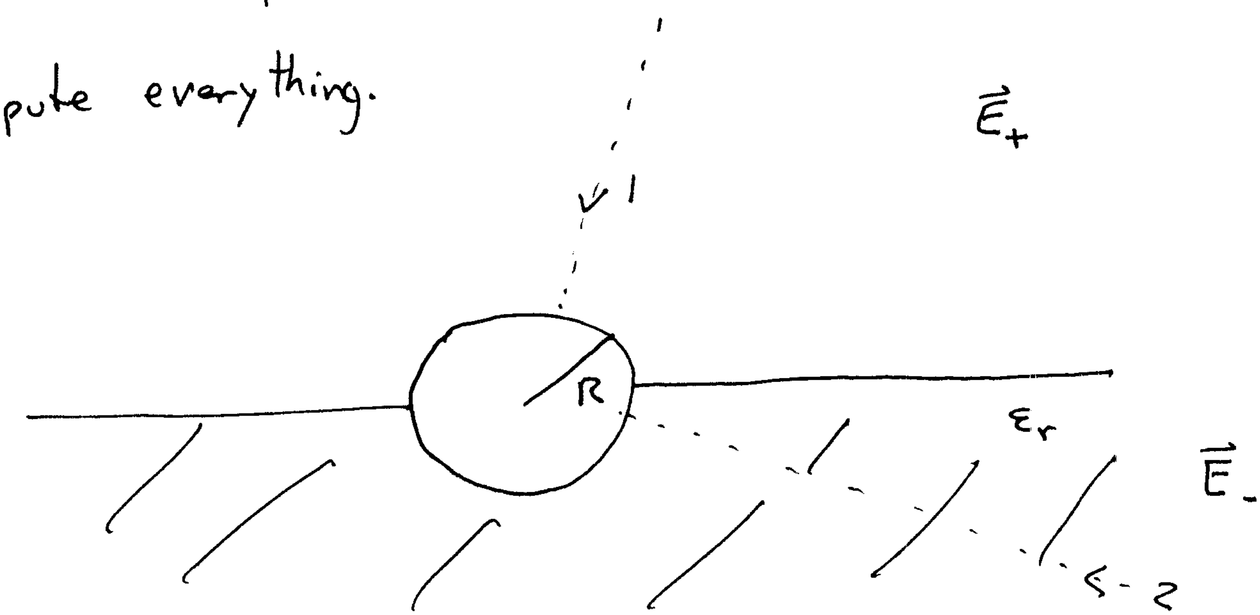
Ex A conducting sphere of radius R is embedded in a dielectric half-plane. The sphere is a potential V_0 with respect to ground. Compute everything.



Gauss $\quad \nabla \cdot \vec{D} = \rho_f$ but we don't know what ρ_f is at the conductor's surface.

We do know $\Delta V_1 = \Delta V_2 = V_0$

$$\Rightarrow -\int \vec{E}_+ \cdot d\vec{l} = -\int \vec{E}_- \cdot d\vec{l} = V_0$$

$$\Rightarrow \vec{E}_+ = \vec{E}_-$$

Now we need fields that satisfy this condition. Let's try the easiest fields to find and see if the work.

If the dielectric wasn't present, the field would be

$$\vec{E}_+ = \frac{kQ_+}{r^2}\hat{r}$$

and $$V_+(R) = \frac{kQ_+}{r}$$

therefore $$V_+(R) = \frac{kQ_+}{R} = V_0$$

and $$Q_+ = \frac{RV_0}{k} = 4\pi\epsilon_0 RV_0$$

$$\Longrightarrow \quad \vec{E}_+ = \frac{RV_0}{r^2}\hat{r}$$

Now suppose the dielectric completely filled the universe.

$$\vec{E}_- = \frac{kQ_-}{\epsilon_r r^2}\hat{r} \qquad V_- = \frac{kQ_-}{\epsilon_r r}$$

$$V_-(R) = \frac{kQ_-}{\varepsilon_r R} = V_0$$

$$Q_- = \frac{\varepsilon_r R V_0}{k}$$

$$\vec{E}_- = \frac{RV_0}{r^2}\hat{r}$$

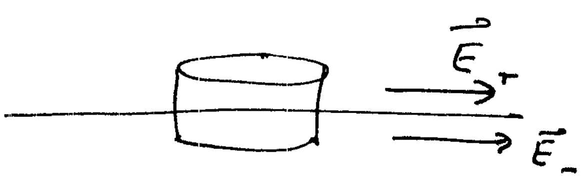
Check Boundary between dielectric and air.

$$\sigma_b = \vec{P}\cdot\hat{n} = \chi_e \varepsilon_0 \vec{E}_- \cdot \hat{n} = 0$$

Boundary Conditions

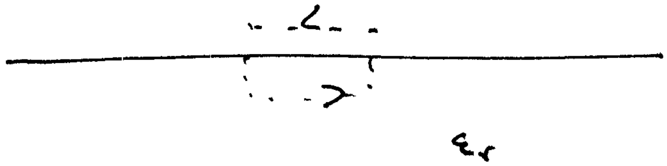
$$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \qquad \nabla\times\vec{E} = 0$$

Using pill box at surface



$$\Phi_e = 0 = \frac{A\sigma_b}{\varepsilon_0} \checkmark$$

Stokesian Loop



$$\oint \vec{E} \cdot d\vec{l} = 0$$

$$= E_- l - E_+ l = 0$$

✓

We have fields that solve the system in each region and match at the boundary, so they are the real fields.

Find the free and bound charge densities

The Q_+ and Q_- we calculated earlier were free charges

so evidently

$$\sigma_+ = \frac{Q_+}{4\pi R^2} \qquad \sigma_- = \frac{Q_-}{4\pi R^2}$$

The total charge on the sphere is

$$Q = \frac{1}{2} Q_+ + \frac{1}{2} Q_-$$

The capacitance of the ~~spher~~ sphere is then

$$C = \frac{Q}{V_0} = \frac{1}{2V_0}\left(Q_+ + Q_-\right)$$

$$= \frac{1}{2V_0}\left(\frac{RV_0}{k} + \frac{\varepsilon_r R V_0}{k}\right)$$

$$= \frac{(1+\varepsilon_r)}{2k} R = 4\pi\varepsilon_0 R\left(\frac{1+\varepsilon_r}{2}\right)$$

We should be able to get the same result by adding the capacitances of capacitance. Let C_+ and C_- be the capacitances of the half-spaces which are connected in parallel in parallel (same ΔV).

$$C_+ = \frac{1}{2}\left(4\pi\varepsilon_0 R\right)$$

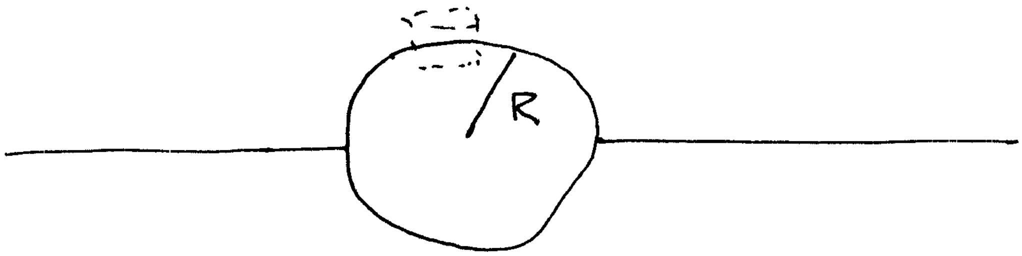
where $4\pi\varepsilon_0 R$ is the capacitance of an isolated sphere.

$$C_- = \frac{1}{2} 4\pi\varepsilon_0 \varepsilon_r R$$

Capacitances in parallel add

$$C = C_+ + C_- = 4\pi\varepsilon_0 R\left(\frac{1+\varepsilon_r}{2}\right)$$ ✓

Back to computing the charges



Method I Use Gaussian pillbox at surface

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \implies \oint \vec{E} \cdot d\vec{a} = \Phi_e = \frac{Q_{enc}}{\varepsilon_0}$$

$$\Phi_e = E(R)A - 0 = \frac{\sigma_t A}{\varepsilon_0}$$

$$\sigma_t = \varepsilon_0 E(R) \qquad \text{where } \sigma_t = \sigma_f + \sigma_b$$

$$\implies \sigma_+ = \varepsilon_0 E_+(R) = \varepsilon_0 \frac{RV_0}{R^2} = \frac{\varepsilon_0 V}{R}$$

since $\sigma_{+b} = 0$

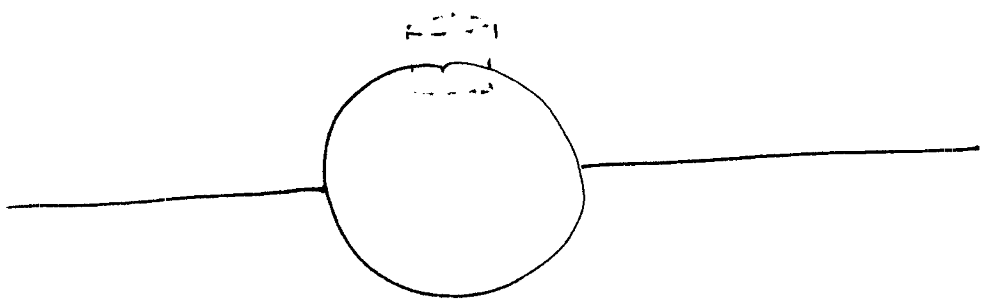
$$\implies \sigma_{-t} = \varepsilon_0 E_-(R) = \frac{\varepsilon_0 V}{R}$$

Now the bound charge isn't zero

Find the Displacement Fields

$$\vec{D}_+ = \varepsilon_0 E \qquad \vec{D}_- = \varepsilon_r \varepsilon_0 \vec{E}$$

$$\nabla \cdot \vec{D} = \rho_f \quad \Rightarrow \quad \int \vec{D} \cdot d\vec{a} = Q_f$$



Pillbox

$$D_+(R)\,A - 0 = \sigma_{f+} A$$

$$\sigma_{f+} = \sigma_f = \varepsilon_0 E(R) \checkmark$$

$$D_-(R) A = \sigma_{f-} A$$

$$\sigma_{f-} = D_-(R) = \varepsilon_0 \varepsilon_r E(R) = \varepsilon_0 \varepsilon_r \frac{R V_0}{R^2}$$

$$= \varepsilon_0 \frac{\varepsilon_r V_0}{R}$$

Therefore the bound charge is

$$\sigma_{t-} = \sigma_{f-} + \sigma_{b-}$$

$$\sigma_{b-} = \sigma_{t-} - \sigma_{f-} = \frac{\varepsilon_0 V}{R} - \frac{\varepsilon_0 \varepsilon_r V_0}{R}$$

$$= \frac{\varepsilon_0 V_0}{R}\left(1 - \varepsilon_r\right)$$

$$\sigma_{b-} = -\chi_e \frac{\varepsilon_0 V_0}{R}$$

Let's check with the polarization

$$\vec{P}_- = \varepsilon_0 \chi_e \vec{E}_- = \varepsilon_0 \chi_e \frac{V_0 R}{r^2}\hat{r}$$

At the sphere, the outward normal is $-\hat{r}$ so the bound charge is

$$\sigma_b = \vec{P}\cdot\hat{n} = -\varepsilon_0 \chi_e \frac{V_0}{R}$$

Is there any volume bound charge?

$$\rho_b = -\nabla \cdot \vec{P} = -\nabla \cdot \left(\frac{\varepsilon_0 \chi_e V_0 R}{r^2} \right) \hat{r}$$

$$= 0$$

Energy -

<u>Method I</u> $\quad U = \frac{1}{2} C V_0^2$

<u>Method II</u> $\quad U = \int \frac{1}{2} \varepsilon_0 E^2 \, dv$

$$= \frac{1}{2} \varepsilon_0 \int_R^\infty 4\pi r^2 \, dr \left(\frac{V_0^2 R^2}{r^4} \right)$$

$$= \frac{1}{2} \varepsilon_0 V_0^2 R^2 \int_R^\infty \frac{dr}{r^2}$$

$$= \frac{1}{2} \varepsilon_0 \frac{V_0^2 R^2}{R} = \frac{4\pi}{2} \varepsilon_0 V_0^2 R$$

But this can't be correct, since it is exactly what we would get without the dielectric

$$C = 4\pi\epsilon_0 R$$

$$U = \frac{1}{2} C V_0^2 = 2\pi\epsilon_0 R V_0^2$$

We left out the energy elastically stored in the dielectric.

$$U = \int \frac{1}{2} \vec{E} \cdot \vec{D} \, dv$$

$= \frac{1}{2} C V_0^2$ as above using $\vec{D}$ calculated earlier.

Now what happens if the dielectric is slightly conducting? Can we get away with the same fields.

$$\vec{J}_f = \sigma \vec{E}$$

$\nabla \cdot \vec{J}_f = 0$ (since $\frac{\hat{r}}{r^2}$ again)

so $\frac{\partial \rho}{\partial t} = 0 \implies$ No free charge develops so our fields are good.

Compute the resistance for the system

$$I = \int_{-} \vec{J} \cdot d\vec{a}$$

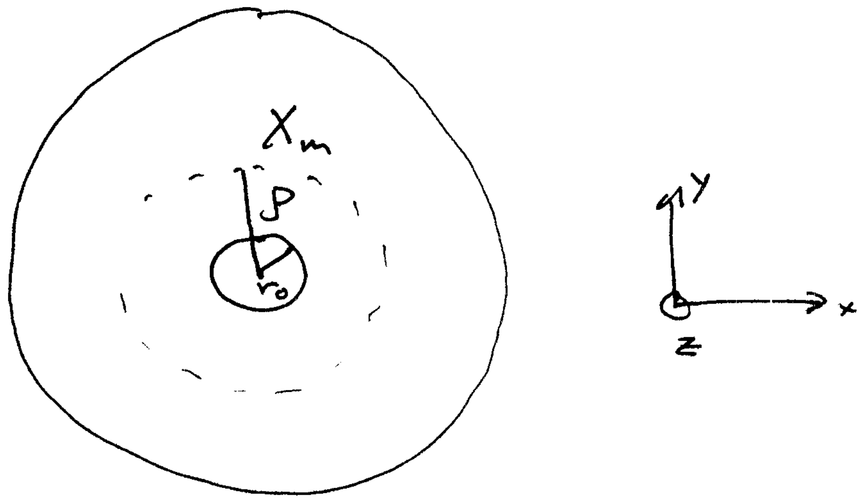
$$= \underbrace{2\pi R^2}_{\text{half-sphere}} \cdot J(R) = 2\pi R^2 \sigma E_{-}(R)$$

$$= 2\pi R^2 \sigma \frac{V_0 R}{R^2}$$

$$= 2\pi R \sigma V_0$$

$$R = \frac{V}{I} = \frac{V_0}{2\pi R \sigma V_0} = \frac{1}{2\pi R \sigma}$$

A long wire is embedded in a ~~dielectric~~ magnetic medium with susceptibility χ_m. Compute $\vec{M}$, $\vec{H}$, $\vec{B}$, $\vec{J}_b$, and $\vec{A}$ everywhere. Let the wire have radius a.



<u>Ampere's Law</u>

$$\nabla \times \vec{H} = J_f \quad \Rightarrow \quad \oint \vec{H} \cdot d\vec{l} = I_f$$

$$2\pi\rho \, H = I_f$$

$$\vec{H} = \frac{I_f}{2\pi\rho}\hat{\phi}$$

Giving χ_m implies the medium is linear.

$$\vec{M} = \chi_m \vec{H}$$

$$= \frac{\chi_m I_f}{2\pi \rho} \hat{\phi} \qquad \mu_r = (1 + \chi_m)$$

$$\vec{B} = \mu_0 \mu_r \vec{H} = \frac{\mu_0 \mu_r I_f}{2\pi \rho} \hat{\phi}$$

<u>Bound Current Density</u>

$$\vec{J}_b = \nabla \times \vec{M}$$

$$= \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} \rho \left(\frac{\chi_m I_f}{2\pi \rho} \right) \right) \hat{z} = 0$$

$$\vec{K}_b = \vec{M} \times \hat{n}$$

At the wire, $\hat{n} = -\hat{\rho}$

$$\vec{K}_b = \left(\frac{\chi_m I_f}{2\pi \rho_0} \hat{\phi} \right) \times (-\hat{\rho}) = \frac{\chi_m I_f}{2\pi r_0} \hat{z}$$

The total bound current is

$$2\pi r_o \vec{K}_b = \chi_m I_f$$

and therefore the total current is

$$I_t = (1 + \chi_m) I_f$$

<u>Vector Potential</u>

$$\vec{B} = \nabla \times \vec{A} \qquad \oint \vec{A} \cdot d\vec{l} = \Phi_m$$

with a normal definition of $\vec{A}(\infty) = 0$, $\vec{A} = \infty$.

Define $\vec{A}(r_o) = 0$

$$\vec{B} = \frac{\mu_0 \mu_r I_f}{2\pi\rho} \hat{\phi} = \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right)$$

Since $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{r''} dv'$

$$\Rightarrow \vec{A} \parallel \hat{z}$$

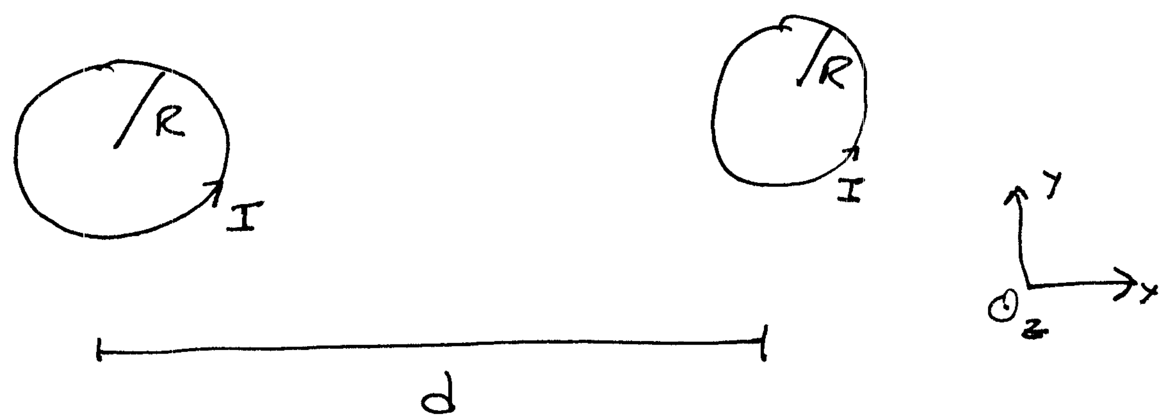
$$-\frac{\partial A_z}{\partial \rho} = \frac{\mu_0 \mu_r I_f}{2\pi \rho}$$

$$A_z = \frac{-\mu_0 \mu_r I_f}{2\pi} \ln \rho + C$$

$$A_z(r_0) = 0 \Rightarrow \quad C = \frac{\mu_0 \mu_r I}{2\pi} \ln r_0$$

$$\vec{A} = \frac{\mu_0 \mu_r I_f}{2\pi} \ln\left(\frac{r_0}{\rho}\right) \hat{z}$$

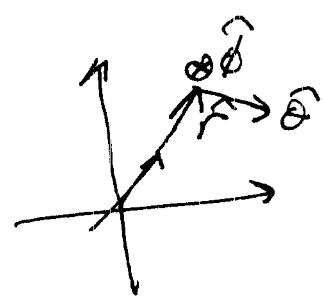
Let's continue to work on our loop. Compute torque and mutual inductance with an identical but well separated loop. Both loops carry a current I. What if the second loop has normal parallel to the surface of paper?



<u>Sln</u> If well separated, use dipole approximation for field.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

$$\vec{m} = I \pi R^2 \hat{z} \equiv m_0 \hat{z}$$



$$\hat{z} = \cos\theta \, \hat{r} - \sin\theta \, \hat{\theta}$$

$$\hat{z} \times \hat{r} = -\sin\theta \, \hat{\theta} \times \hat{r}$$

$$= -\sin\theta \, \hat{\phi}$$

$$\vec{A}(\vec{r}) = \frac{-\mu_0 m_0}{4\pi} \frac{\sin\theta\,\hat{\phi}}{r^2}$$

$$\frac{\nabla \times \vec{A}}{\left(-\frac{\mu_0 m_0}{4\pi}\right)} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\frac{\sin\theta^2}{r^2}\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\sin\theta}{r^2}\right)\hat{\theta}$$

$$= \frac{1}{r^3\sin\theta}\, 2\sin\theta\cos\theta\,\hat{r} + \frac{1}{r^3}\sin\theta\,\hat{\theta}$$

$$= \frac{1}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\vec{B} = \frac{-\mu_0 m_0}{4\pi r^2}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Evaluate at $(d, 0, 0) \rightarrow r = d,\ \theta = \pi/2$

$$\vec{B} = -\frac{\mu_0 m_0}{4\pi d^2}\,\hat{z}$$

Flux through second loop

$$\phi_2 = \pi R^2 B = \frac{-\mu_0 (\pi R^2)^2 I}{4\pi d^2}$$

$$M_{21} = \frac{\phi_2}{I} = \frac{\mu_0 (\pi R^2)^2}{4\pi d^2}$$

Torque

$$\vec{\tau} = \vec{m}_2 \times \vec{B}$$

$$\vec{m}_2 = \pi R^2 I \hat{z}$$

$$\vec{\tau} = 0$$

If

then $M_{21} = 0$ and

$$\tau = \vec{m} \times B$$

$$= (\pi R^2 I)\left(\frac{\mu_0 \pi R^2 I}{4\pi d^2}\right)(\hat{x} \times -\hat{z})$$

$$= \frac{\mu_0}{4\pi d^2}(\pi R^2 I)^2 \hat{y}$$

What can you do?

Force + Torque

$$\vec{F} = q\vec{v}\times\vec{B}$$

$$d\vec{F} = I\,d\vec{l}\times\vec{B}$$

$$\vec{F} = \nabla(\vec{m}\cdot\vec{B})$$

$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E}$$

$$\vec{T} = \vec{m}\times\vec{B} \qquad \vec{T} = \vec{p}\times\vec{E}$$

$$U = -\vec{m}\cdot\vec{B} \qquad U = -\vec{p}\cdot\vec{E}$$

$$\vec{F} = -\frac{dW}{dx}$$

Can be applied to any system where a dipole moment can be obtained.

$$U = \tfrac{1}{2}LI^2$$

$$U = \tfrac{1}{2}CV^2$$

Energy

$$u = \tfrac{1}{2}\vec{E}\cdot\vec{D} \quad \text{or} \quad \tfrac{1}{2}\varepsilon_0\vec{E}\cdot\vec{E}$$

$$u = \tfrac{1}{2}\vec{B}\cdot\vec{H} \quad \text{or} \quad \tfrac{1}{2}\frac{\vec{B}\cdot\vec{B}}{\mu_0}$$

$$u = \tfrac{1}{2}\rho V \qquad u = \tfrac{1}{2}\vec{A}\cdot\vec{J}$$

$$W = Q\Delta V \qquad \text{emf} = \int(\vec{E}+\vec{v}\times\vec{B})\cdot d\vec{l}$$

Can be calculated for any system where we have the fields.

Potential

$$V = -\int \vec{E} \cdot d\vec{l}$$

$$\Phi_m = \int \vec{A} \cdot d\vec{l}$$

$$\vec{E} = -\nabla V$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{A} = \int \frac{\mu_0 J}{4\pi r''} dv'$$

$$V = \int \frac{\rho \, dv'}{4\pi\epsilon_0 r''}$$

I. Should be able to calculate potential

① Directly from field by undoing ∇ or $\nabla\times$

② From symmetric field by integrating

③ Directly from current.

Systems

I. Polarization

Ⓐ Electrets -- Polarization given field charge and field $(\vec{E} + \vec{D})$

$$\sigma_b = \hat{n} \cdot \vec{P} \qquad \rho_b = -\nabla \cdot \vec{P}, \ \vec{P}$$

Ⓑ Linear materials - E_0 or Q given field

E_i, σ_b, ρ_b, $\vec{P}$, $\vec{D}$, $\vec{P}$

III Magnetization

(A) Permanent Magnets ($\vec{M}$ given)

J_b, K_b, $\vec{H}$, $\vec{B}$, $\vec{m}$

(B) Linear Magnetic Materials μ_r given

Find field given current or applied field

All above.

IV Conductors σ/ρ given.

Find field, current, resistance, voltage in steady state.

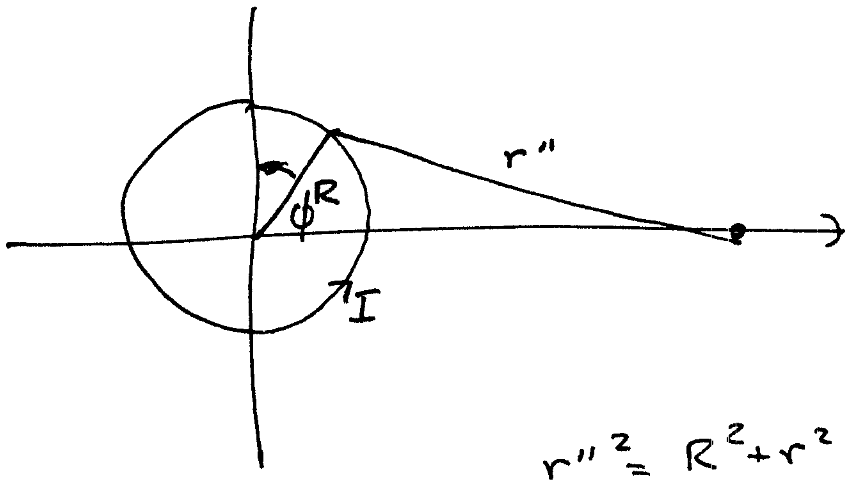
V Dipole fields

$$V(r) = \frac{p}{4\pi\varepsilon_0 r^2} \qquad \vec{A}(r) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

VI Anything from first test

V. Capacitance, Inductance, Mutual Inductance

Ex Ring of Current



$$r''^2 = R^2 + r^2 - 2rR\cos\phi$$

$$\vec{A}(r) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\ell}}{r''}$$

$$d\vec{\ell} = \oint R\,d\phi\,\hat{\phi}$$

$$\hat{\phi} = (-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$

Only

$$\vec{A}(r) = \frac{\mu_0 I \hat{y}}{4\pi} \int_0^{2\pi} \frac{R\,d\phi\cos\phi}{\sqrt{R^2 + r^2 - 2rR\cos\phi}}$$

Evidently, $\hat{y} \rightarrow \hat{\phi}$

$$\vec{A}(\rho) = \frac{\mu_0 I}{4\pi} \frac{R\,\hat{\phi}}{\sqrt{\rho^2+R^2}} \int_0^{2\pi} \frac{\cos\phi \, d\phi}{\sqrt{1 - m^2 \cos\phi}}$$

$$m^2 = \frac{2\rho R}{R^2+\rho^2}$$

<u>Maple Gives Me</u>

$$\vec{A} = \frac{\mu_0 I}{4\pi} \frac{R}{\sqrt{\rho^2+R^2}} \cdot \frac{2(R+\rho)}{R\rho\sqrt{R^2+\rho^2}} \Big[(\rho^2+R^2) K(\gamma) - (\rho+R)^2 E(\gamma) \Big]$$

$$\vec{A} = \frac{\mu_0 I}{2\pi\rho} \frac{R+\rho}{R^2+\rho^2} \Big[(\rho^2+R^2) K(\gamma) - (\rho+R)^2 E(\gamma) \Big]$$

where K and E are elliptical functions

and $\gamma = 2\sqrt{\dfrac{rR}{(r+R)^2}}$

> $int\left(\frac{cos(t)}{(1-a\cdot cos(t))^{\left(\frac{1}{2}\right)}},\ t=0..2\cdot Pi\right);$

$$\frac{1}{\sqrt{\frac{a}{1+a}}\,\sqrt{-a+1}}\left(2\left(\mathrm{EllipticF}\left(\sqrt{2}\,\sqrt{\frac{a}{1+a}},\frac{1}{2}\sqrt{2}\,\sqrt{\frac{1+a}{a}}\right)\right.\right.$$
$$\left.\left.-2\,\mathrm{EllipticE}\left(\sqrt{2}\,\sqrt{\frac{a}{1+a}},\frac{1}{2}\sqrt{2}\,\sqrt{\frac{1+a}{a}}\right)\right)\sqrt{-\frac{a-1}{1+a}}\,\sqrt{2}\right) \tag{1}$$

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